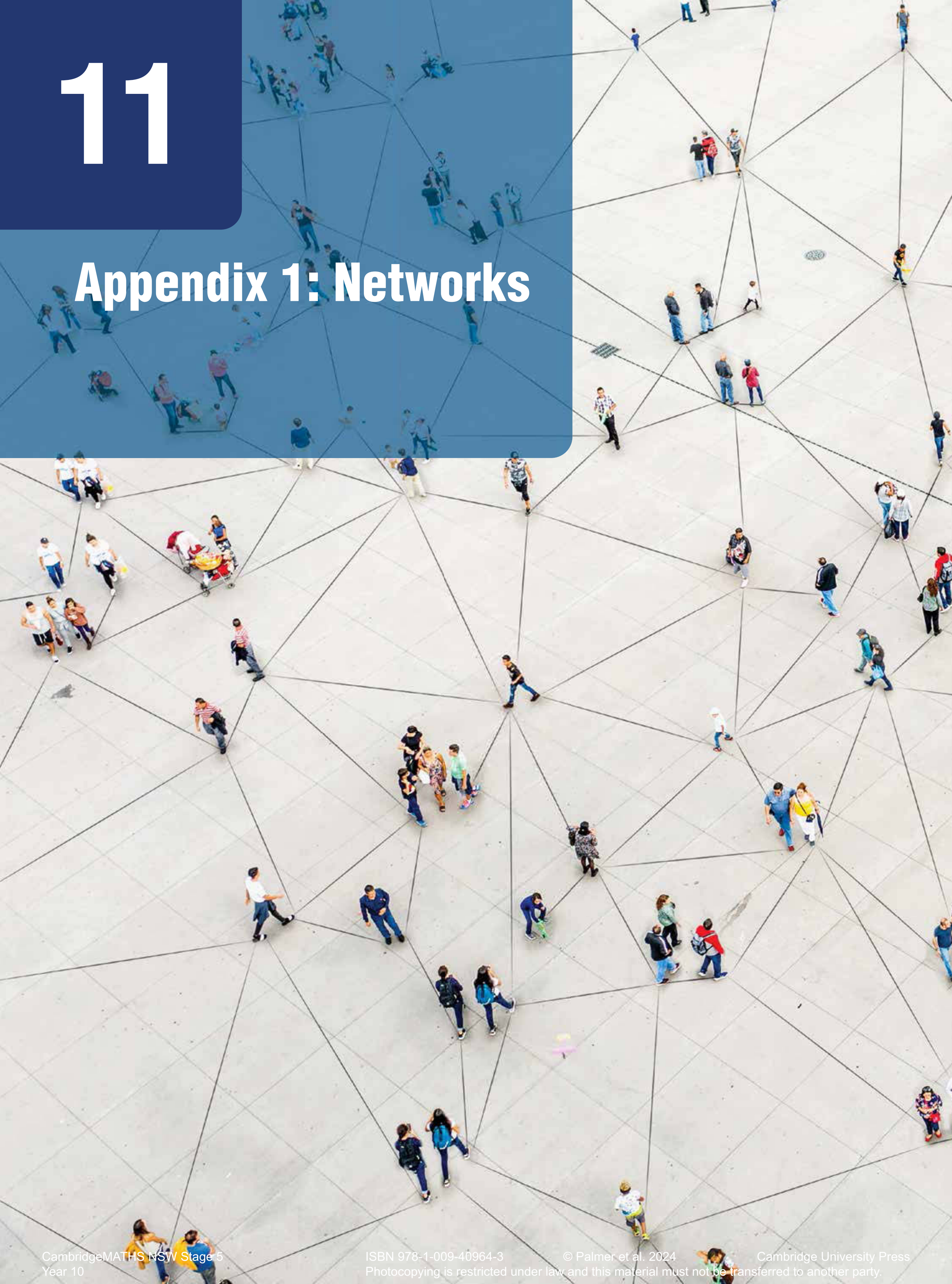
11

Appendix 1: Networks



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Chapter contents

NSW Syllabus

In this chapter, a student:

- solves problems involving the characteristics of graphs/networks, planar graphs and Eulerian trails and circuits (MA5-NET-P-01)

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

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11A Introduction to networks

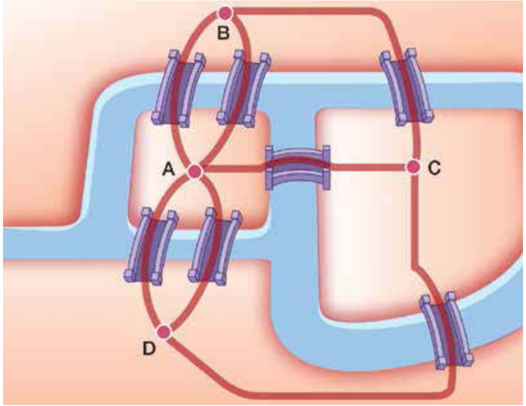
Learning intentions for this section:
- To know what is meant by a network graph
- To know the key features of a network graph
- To be able to find the degree of a vertex and the sum of degrees for a graph
- To be able to describe simple walks through a network using the vertex label

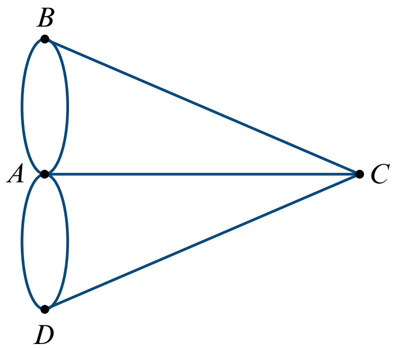
Past, present and future learning:
- This section has virtually no prerequisite knowledge
- It addresses a Path Topic for students progressing to Standard
- It has no relevance to Stage 6 Advanced
- It will be revised and extended in Year 12 Standard

A network is a collection of points (vertices or nodes) which can be connected by lines (edges). Networks are used to help solve a range of real-world problems including travel and distance problems, intelligence and crime problems, computer network problems and even metabolic network problems associated with the human body. In Mathematics, a network diagram can be referred to as a graph, not to be confused with the graph of a function like $y = x^2 + 3$.

Lesson starter: The Konigsberg bridge problem

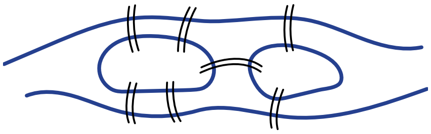
The seven bridges of Konigsberg is a well-known historical problem solved by Leonhard Euler who laid the foundations of graph theory. It involves two islands at the centre of an old German city connected by seven bridges over a river as shown in these diagrams.





The problem states: Is it possible to start at one point and pass over every bridge exactly once and return to your starting point?

- Make a copy of this simplified map of the seven bridges of Konigsberg and try tracing out a walk that crosses all bridges exactly once. Try starting at different places.



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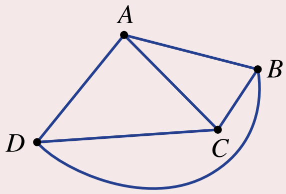
- Investigate if there might be a solution to this problem if one of the bridges is removed.
- Investigate if there might be a solution to this problem if one bridge is added.

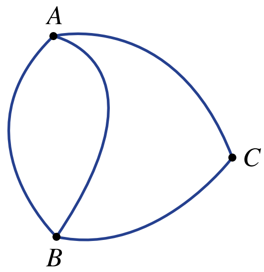
KEY IDEAS

- A **network** or **graph** is a diagram connecting points using lines.
  - The points are called **vertices** (or **nodes**). Vertex is singular, vertices is plural.
  - The lines are called **edges**.
- The **degree** of a vertex is the number of edges connected to it.
  - A vertex is odd if the number of edges connected to it is odd.
  - A vertex is even if the number of edges connected to it is even.
- The sum of degrees is calculated by adding up the degrees of all the vertices in a graph.
  - It is also equal to twice the number of edges.
- A **walk** is any type of route through a network.
  - A walk can be defined using the vertex labels.
  - Example: A-B-C-A-D.



BUILDING UNDERSTANDING

1 Here is a graph representing roads connecting three towns A, B and C.



a How many different roads (edges) does this graph have?

b How many different vertices (nodes) does this graph have?

c If no road (edge) is used more than once and no town (vertex/node) is visited more than once, how many different walks are there if travelling from:

i A to C?

ii A to B?

iii B to C?

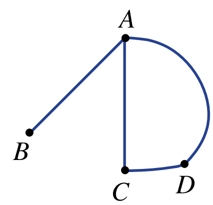
d How many roads connect to:

i town A?

ii town B?

iii town C?

2 This graph uses four edges to connect four vertices.



a How many edges connect to vertex A?

b What is the degree of vertex A?

c State the total number of edges on the graph.

d By finding the number of edges connected to each vertex, find the sum of degrees for the graph.

e What do you notice about the total number of edges and the sum of degrees for this graph?

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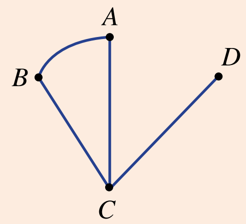
Example 1 Determining features of a graph

For the graph shown complete the following.



a State the total number of:
  i vertices (nodes) **ii** edges.

b State the degree of:
  i vertex A **ii** vertex B
  iii vertex C **iv** vertex D.

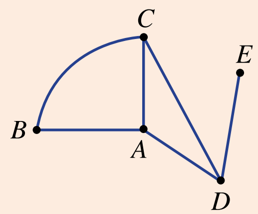
c Find the sum of degrees for the graph.

d State which vertices are:
  i odd **ii** even.

SOLUTION	EXPLANATION
a **i** 4	The four vertices are A, B, C and D.
ii 4	The four edges are AB, AC, BC and CD.
b **i** 2	Simply count how many edges connect to each vertex.
ii 2	
iii 3	
iv 1	
c $2 + 2 + 3 + 1 = 8$	Add up all the degrees from all vertices.
d **i** C and D	The degrees of vertices C and D are 3 and 1.
ii A and B	The degrees of vertices A and B are both 2.

Now you try

For the graph shown complete the following.



a State the total number of:
  i vertices (nodes) **ii** edges.

b State the degree of:
  i vertex A **ii** vertex B
  iii vertex C **iv** vertex D
  v vertex E.

c Find the sum of degrees for the graph.

d State which vertices are:
  i odd **ii** even.

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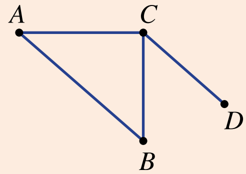
Example 2 Finding a walk

Consider this graph connecting the four vertices A, B, C and D. Without visiting a vertex (node) more than once or using an edge more than once, find how many walks there are connecting the following pairs of vertices.

a A and D

b B and D



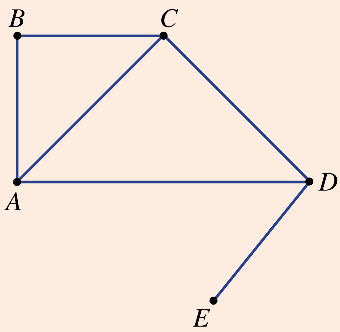
SOLUTION	EXPLANATION
a 2	The two walks are A-B-C-D and A-C-D.
b 2	The two walks are B-A-C-D and B-C-D.

Now you try

Consider this graph connecting the five vertices A, B, C, D and E. Without visiting a vertex more than once or using an edge more than once, find how many walks there are connecting the following pairs of vertices.

a A and E

b B and D

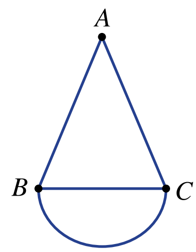


Exercise 11A

FLUENCY	1, 2, 4–6	1, 3–6	3–6

Example 1

1 For the graph shown complete the following.



a State the total number of:

  i vertices (nodes) **ii** edges.

b State the degree of:

  i vertex A **ii** vertex B

  iii vertex C.

c Find the sum of degrees for the graph.

d State which vertices are:

  i odd **ii** even.

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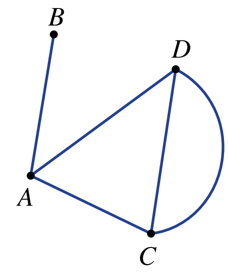
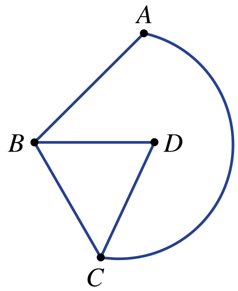
2 For the graph shown complete the following.



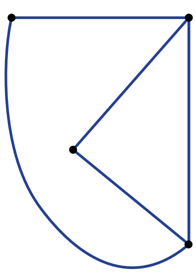
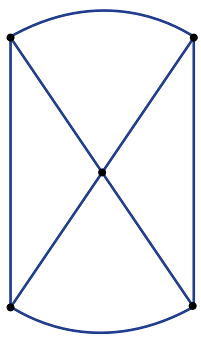
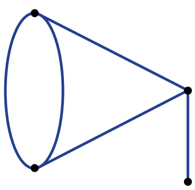
a State the total number of:
   i vertices (nodes)   ii edges.
b State the degree of:
   i vertex A ii vertex B
   iii vertex C iv vertex D.
c Find the sum of degrees for the graph.
d State which vertices are:
   i odd   ii even.

3 For the graph shown complete the following.

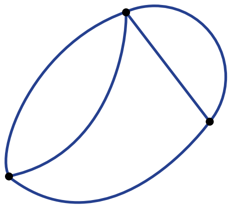
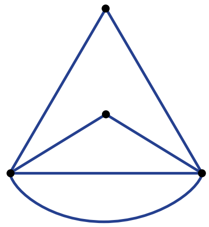
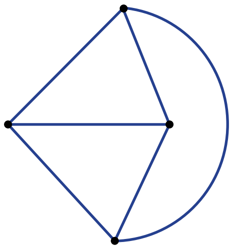


a State the total number of:
   i vertices (nodes)   ii edges.
b State the degree of:
   i vertex A ii vertex B
   iii vertex C iv vertex D.
c Find the sum of degrees for the graph.
d State which vertices are:
   i odd   ii even.

4 Which of the following graphs has vertices which are all odd?

A

B

C


5 Which of the following graphs has vertices which are all even?

A

B

C


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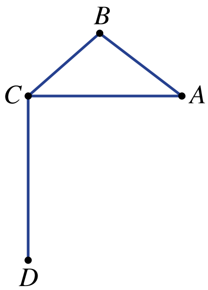
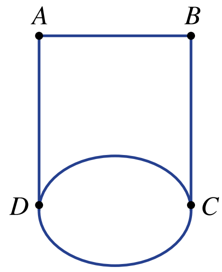
Example 2

6 Consider this graph connecting the four vertices A, B, C and D. Without visiting a vertex (node) more than once or using an edge more than once, find how many walks there are connecting vertex A with vertex D.

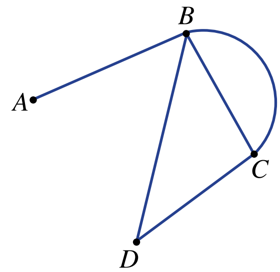
a



b



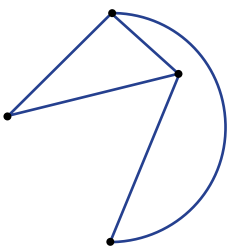
c



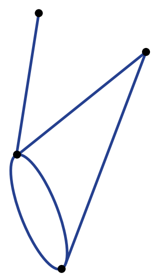
PROBLEM–SOLVING	7, 8	7–9	8–10

7 Which of the following graphs has the greatest sum of degrees?

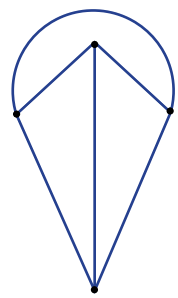
A



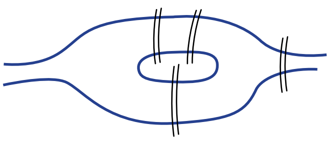
B



C



8 Similar to the Konigsberg bridge problem this diagram shows an island in the middle of a river and four bridges.



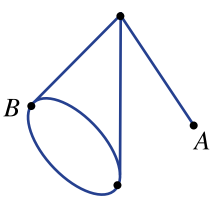
a Is it possible to start on the island and finish on the island by crossing each of the four bridges exactly once?

b Is it possible to start on one side of the river (not on the island) and finish on the same side of the river by crossing each of the four bridges exactly once?

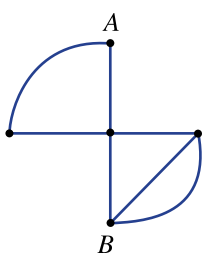
c Is it possible to start on one side of the river (not on the island) and finish on the opposite side of the river by crossing each of the four bridges exactly once?

9 Which of the following graphs has the greatest number of walks connecting vertices A and B if no edge or vertex can be used more than once?

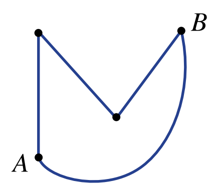
A



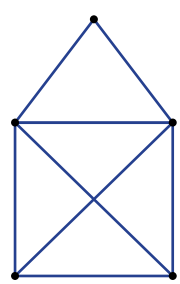
B



C



10 Is it possible to find a walk around this graph so that each edge is used exactly once? There is no restriction on the number of times vertices are visited or where you should start and finish. If the answer is 'yes', draw your walk. The place in the middle of the square is not a vertex, just an intersection of two edges. You cannot change direction at this position.



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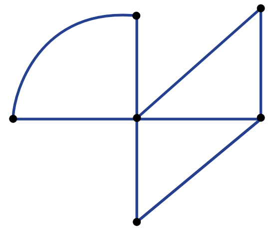
REASONING	11	11, 12	12, 13

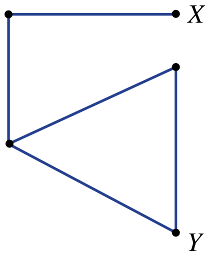
11 Consider the given graph.



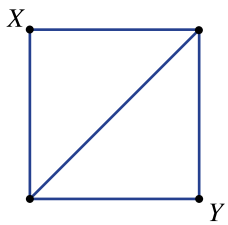
- **a** How many edges does the graph have?
- **b** Find the sum of degrees for the graph.
- **c** What do you notice about your answers to part **a** and part **b** above? Decide if your conclusion is true for other graphs by drawing your own.

12 Decide if it is possible to find a walk from X to Y in these graphs by visiting each vertex exactly once and using each edge exactly once.

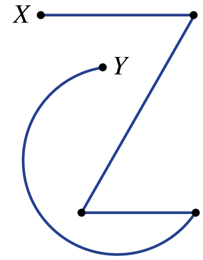
a



b

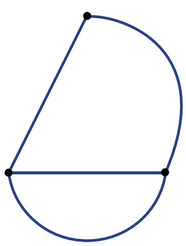


c

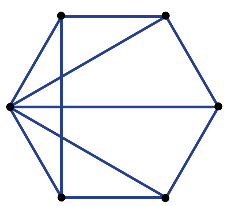


13 Sometimes edges can be removed from a graph with all remaining vertices still connected to the graph. Find the maximum number of edges that can be removed from these graphs so that all vertices remain connected to the graph.

a

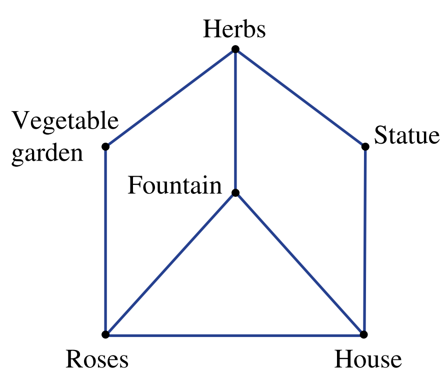


b



ENRICHMENT: Garden tours with conditions	–	–	14

14 Consider this graph representing a garden including six key features connected with garden paths.



- **a** How many different ways are there of walking from the House to the Vegetable garden without visiting a feature more than once or using a garden path more than once?
- **b** How many different tours are possible starting and ending at the House? Other garden features can only be visited once but the tour does not need to visit all features. Do not count tours which are the same if you ignore direction.

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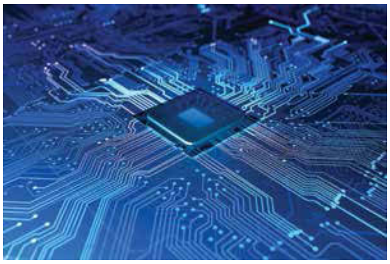
11B Isomorphic and planar graphs

Learning intentions for this section:
- To know what isomorphic and planar graphs are
- To be able to identify isomorphic and planar graphs
- To be able to verify Euler's formula and use it to determine features of a graph

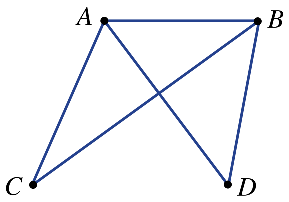
Past, present and future learning:
- This section has virtually no prerequisite knowledge
- It addresses a Path Topic for students progressing to Standard
- It has no relevance to Stage 6 Advanced
- It will be revised and extended in Year 12 Standard

Many of the graphs considered so far may look different but in fact contain exactly the same information. Such graphs are said to be isomorphic and could be redrawn to look the same. Further, most of the graphs previously considered have been drawn with no intersecting edges. Such graphs are called planar graphs and often occur naturally in problems involving electronic circuits, railway networks and utility lines.



Lesson starter: Exploring Euler's formula with planar graphs

Consider this graph representing the connection between four vertices. There is no vertex at the intersection of the edges AD and BC.



- Is it possible to redraw the graph so that all the vertices and edges are retained but edges do not intersect? Try this with a new drawing.
- Your new drawing should be a planar graph with no intersecting edges. How many different regions make up the graph including the outside region? These are called faces.
- Calculate the following.
  - $v + f$ (The sum of the number of vertices and the number of faces)
  - $e + 2$ (Two more than the number of edges)
- What do you notice about the two calculations above?
- Try drawing a different planar graph to verify your conclusion.

KEY IDEAS

- **Isomorphic graphs** contain the same information including:
  - the same number of vertices
  - the same number of edges
  - the same edge connections.

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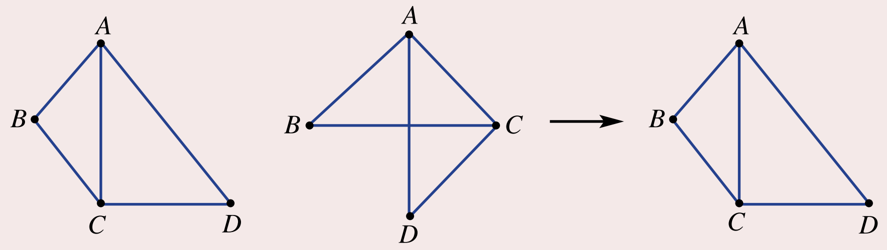
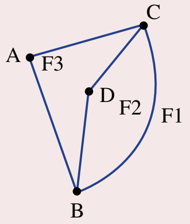
- Isomorphic graphs can be drawn to look the same such as Graph A and Graph B below.



- **A planar graph** can be drawn so that it has no intersecting edges.
  - Many graphs may not look planar initially but could be redrawn without any intersecting edges.
  - A non-planar graph cannot be drawn without some edges crossing.
- A **face** of a graph is a region bound by a set of edges and vertices.
- **Euler's formula** applies to planar graphs and is such that $v + f = e + 2$ where:
  - v is the number of vertices
  - f is the number of faces including the outside face
  - e is the number of edges.
  - Euler's formula is often also written as $v - e + f = 2$.

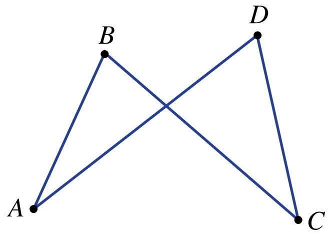


BUILDING UNDERSTANDING

1 Consider this graph connecting the four vertices A, B, C and D.

a Try redrawing the graph with no edges crossing.

b Would you therefore say that the graph is planar?



2 Consider this planar graph.

a State the number of:

i vertices, v

ii edges, e

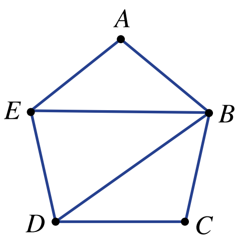
iii faces, f. (Don't forget to count the outside face)

b Calculate the following.

i $v + f$

ii $e + 2$

c What do you notice about your answers to part **b**?



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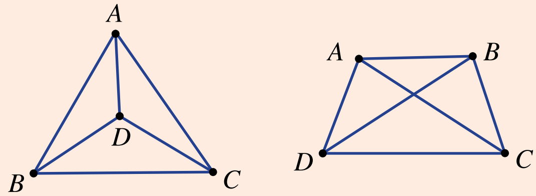
Example 3 Deciding if graphs are isomorphic

Decide if the following two graphs are isomorphic.



SOLUTION

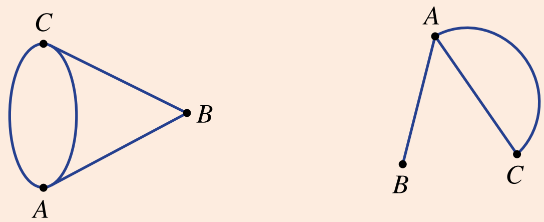
Yes, the graphs are isomorphic.

EXPLANATION

Both graphs have the same number of vertices and edges and the same connections. They could be redrawn to look identical.

Now you try

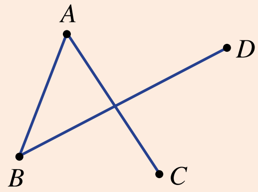
Decide if the following two graphs are isomorphic.



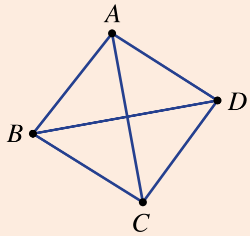
Example 4 Deciding if graphs are planar

Decide if the following graphs are planar or non-planar.

a



b

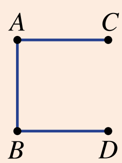


SOLUTION

a Yes, the graph is planar.

EXPLANATION

The graph can be drawn without any intersecting edges.



Continued on next page

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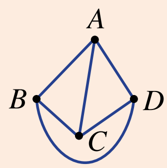
b Yes, the graph is planar.

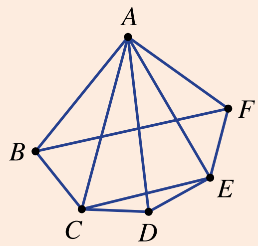
The graph can be drawn without any intersecting edges.



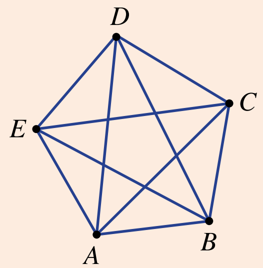
Now you try

Decide if the following graphs are planar or non-planar.

a

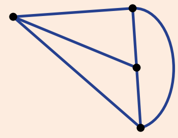


b



Example 5 Verifying Euler's formula

Consider this planar graph.



a Find the number of:

- **i** vertices, v
- **ii** edges, e
- **iii** faces, f.

b Verify Euler's formula using the information in the given planar graph.

SOLUTION

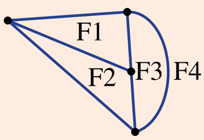
a **i** 4

ii 6

iii 4

EXPLANATION

Simply count the number of vertices, edges and faces. Don't forget to count the infinite, outside face.



SOLUTION

b $v + f = 4 + 4 = 8$

$e + 2 = 6 + 2 = 8$

OR $v - e + f = 4 - 6 + 4 = 2$

Therefore, Euler's formula is verified.

EXPLANATION

Substitute the values of v, e, and f into Euler's formula and check that the left-hand side equals the right-hand side. Using $v + f = e + 2$ is equivalent to using $v - e + f = 2$.

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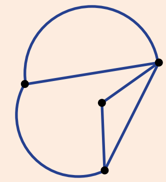
Now you try



Consider this planar graph.

a Find the number of:

  i vertices, v

  ii edges, e

  iii faces, f.

b Verify Euler's formula using the information in the given planar graph.

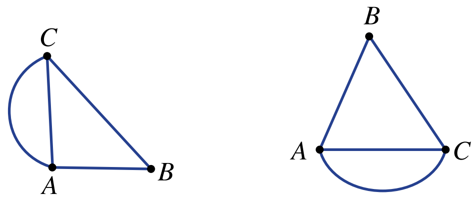
Exercise 11B

FLUENCY	1–3	1–4	1, 2, 4

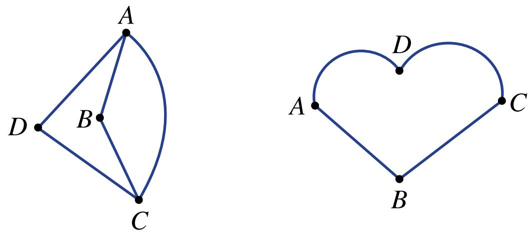
Example 3

1 Decide if the following pairs of graphs are isomorphic.

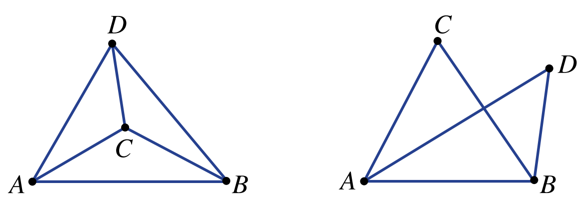
a



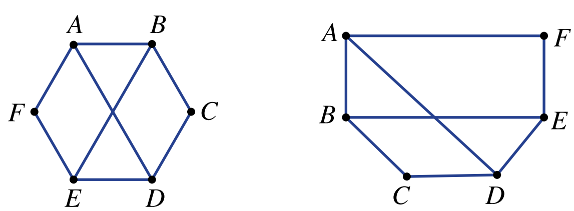
b



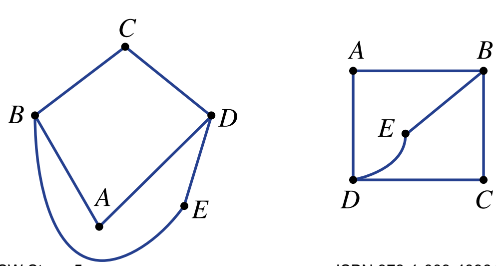
c



d



e



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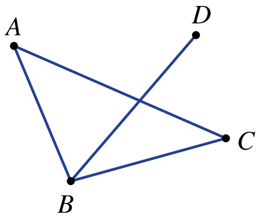
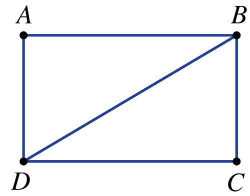
Example 4

2 Decide if the following graphs are planar or non-planar.

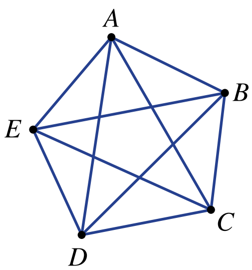
a



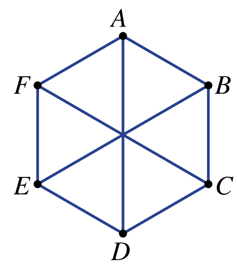
b



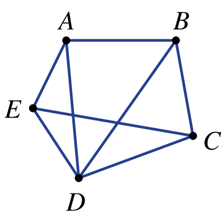
c



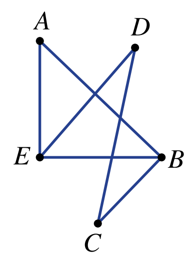
d



e

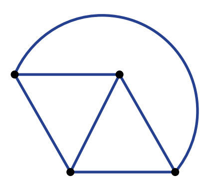


f



Example 5

3 Consider this planar graph.



a Find the number of:

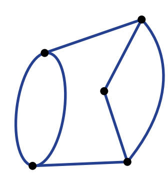
i vertices, v

ii edges, e

iii faces, f.

b Verify Euler's formula using the information in the given planar graph.

4 Consider this planar graph.



a Find the number of:

i vertices, v

ii edges, e

iii faces, f.

b Verify Euler's formula using the information in the given planar graph.

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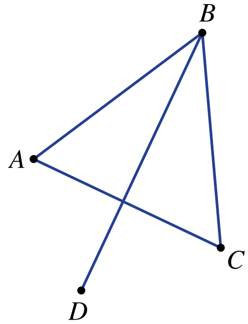
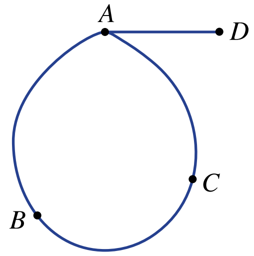
PROBLEM-SOLVING	5, 7	6, 7	7, 8

5 Which pair of the three graphs, A, B or C are isomorphic?

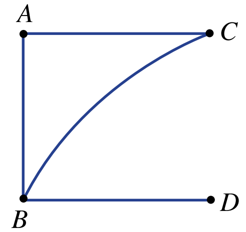
A



B

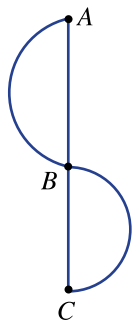


C

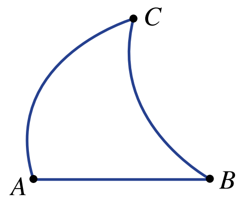


6 Which pair of the three graphs, A, B or C are isomorphic?

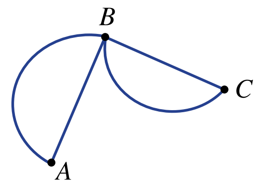
A



B

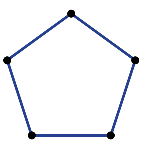


C

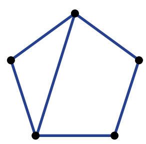


7 The following pentagons include some diagonal edges. Decide which ones are planar.

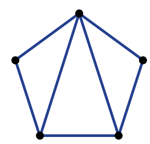
A



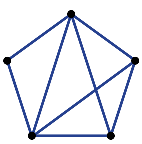
B



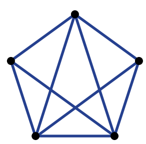
C



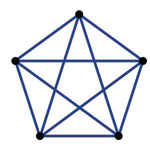
D



E



F



8 Use Euler's formula to answer the following which involve planar graphs.

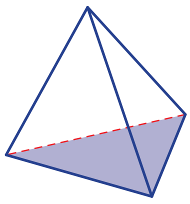
a If the number of vertices is 6 and the number of faces is 8, then find the number of edges.

b If the number of vertices is 8 and the number of edges is 12, then find the number of faces.

c If the number of edges is 10 and the number of faces is 8, then find the number of vertices.

REASONING	9	9, 10	10, 11

9 Consider this four-sided, 3-dimensional polyhedron called a tetrahedron.



a State the number of vertices, edges and faces.

b Verify Euler's formula for the tetrahedron.

c Now draw the tetrahedron as a 2-dimensional planar graph. It does not need to look like a tetrahedron, just a planar graph with the same vertex, edge and face information.

d Verify Euler's formula using your planar graph from part **c**.

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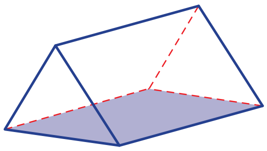
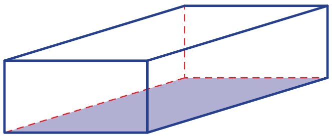
10 Complete parts **a**–**d** from question **9** above for the following 3-dimensional polyhedra.

a Pentahedron



b Hexahedron



11 An octahedron is a 3-dimensional polyhedron with eight faces. By drawing both a planar graph in two dimensions and a 3-dimensional representation of the solid, verify Euler's formula.

ENRICHMENT: Platonic solids rule	–	–	12

12 Earlier in the exercise we considered polyhedra as planar graphs which satisfy Euler's formula. Another interesting rule which only applies to regular polyhedra, the Platonic solids, is $\frac{1}{n} + \frac{1}{d} = \frac{1}{2} + \frac{1}{e}$ where:

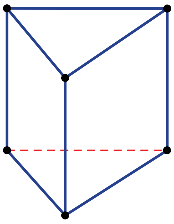
n is the number of sides on each face
e is the number of edges
d is the number of edges joining each vertex. This will be the same for every vertex.

Note that there only five Platonic solids including the regular tetrahedron (4 faces), cube (6 faces), octahedron (8 faces), dodecahedron (12 faces) and icosahedron (20 faces). Each face is a regular polygon.

a Consider the first Platonic solid, the regular tetrahedron.
- **i** Find the values of n, e and d.
- **ii** Verify the formula $\frac{1}{n} + \frac{1}{d} = \frac{1}{2} + \frac{1}{e}$.

b Repeat part **a** for one or more of the other Platonic solids.

c Consider this pentahedron. Show that Euler's formula can be verified and explain why the rule $\frac{1}{n} + \frac{1}{d} = \frac{1}{2} + \frac{1}{e}$ cannot.



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11C Trails, paths and Eulerian circuits

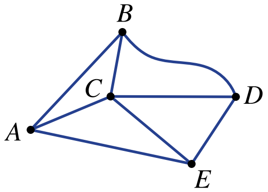
Learning intentions for this section:
- To know the difference between a walk, trail, circuit, path and cycle in a network
- To know what Eulerian trails and cycles are
- To be able to identify different types of walks within a network
- To be able to find Eulerian trails and circuits within a network

Past, present and future learning:
- This section has virtually no prerequisite knowledge
- It addresses a Path Topic for students progressing to Standard
- It has no relevance to Stage 6 Advanced
- It will be revised and extended in Year 12 Standard

Depending on the application of graph theory, the desired walk through a network may involve a wide range of factors. The number of times that a vertex can be visited, or an edge is used, may or may not be important. In one application, it may be a requirement that a walk starts and ends at the same vertex. Alternatively, each edge might have to be used exactly once. In this section we will define and explore the different types of walks through a network and focus on a special group of them called Eulerian trails.

Lesson starter: Exploring Buxton

The small town of Buxton has five key tourist destinations labelled here with the letters A, B, C, D and E. They are connected by footpaths as shown in this network diagram.



A particular trail is defined by B-C-D-E-C.

- Are there any footpaths (edges) used more than once in this trail?
- Are there any destinations (vertices) visited more than once?
- What extensions to the trail could be added (at the end) so that it becomes a circuit where it ends at where it began? Is it possible to do this without using a particular footpath more than once?

A path is defined by A-B-C-E.

- Are there any footpaths (edges) used more than once in this trail?
- Are there any destinations (vertices) visited more than once?
- What extensions to the path could be added (at the end) so that it becomes a cycle where it ends at where it began? Is it possible to do this without using a particular footpath (edge) or destination (vertex) more than once?

Now imagine trying to visit every destination in Buxton.

- Is it possible to visit all the destinations on a given path so that all destinations are visited exactly once and no footpath is used more than once? If so, how?

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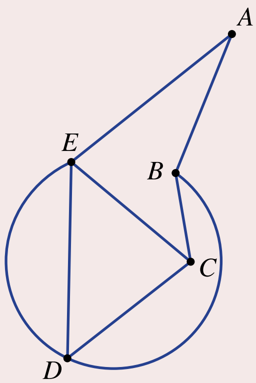
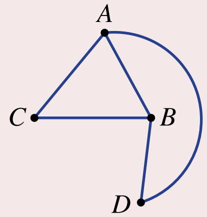
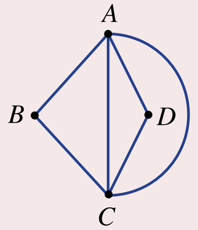
KEY IDEAS



- A **walk** is any type of route through a network. Examples:
  - A-B-D-C-B
  - E-D-B-C-D-E
- A **trail** is a walk where edges are not repeated. Examples:
  - A-E-D-C-E
  - C-E-D-C-B
- A **circuit** is a trail that begins and ends at the same vertex. Examples:
  - E-D-B-C-D-E
  - B-C-D-E-D-B
- A **path** is a walk where vertices and edges are not repeated. Examples:
  - E-D-C-B
  - B-C-D-E-A
- A **cycle** is a path that begins and ends at the same place. Examples:
  - E-D-C-E
  - A-E-D-B-A
- An **Eulerian trail** is a walk where every edge is included exactly once. Vertices are allowed to be revisited.
  - Eulerian trails exist if there are zero or exactly two vertices of odd degree.
  - If exactly two vertices are of odd degree, then Eulerian trails start at one of these vertices and end at the other.
  - If there are zero vertices of odd degree, then all Eulerian trails are circuits.
  - This graph has two vertices of odd degree and an example of an Eulerian trail is B-D-A-C-B-A.



- An **Eulerian circuit** is an Eulerian trail which starts and ends at the same vertex.
  - An Eulerian circuit exists if and only if zero vertices are of odd degree.
  - This graph has zero vertices of odd degree and an example of an Eulerian circuit is A-B-C-A-D-C-A.
  - An Eulerian circuit can start at any vertex.



BUILDING UNDERSTANDING

1 Give the formal name (walk, trail, circuit, path or cycle) of each of the following.

a Any type of route through a network.
b A trail that begins and ends at the same vertex.
c A path that begins and ends at the same vertex.
d A walk where edges are only used once.
e A walk where vertices and edges are only used once.

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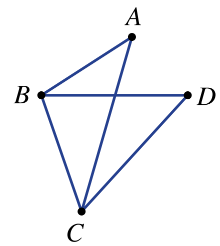
2 Consider this simple graph.



a Which vertices are of odd degree?

b Find a trail so that all edges are used exactly once. Write down your trail using the letters A, B, C and/or D.

c What is the name of the type of walk you discovered in part b?

Assume now that the edge AD is added to the graph.

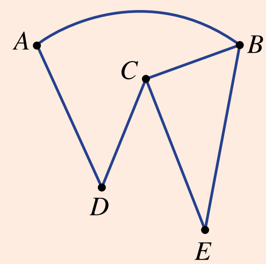
d Is it now possible to find an Eulerian trail through the network?

e How many vertices are of odd degree?

f How many vertices need to be of odd degree if a trail is to be Eulerian?

Example 6 Defining walks

Consider this graph.



a Decide if the following walks are trails. Answer Yes or No.
 i A-D-C-B ii A-B-C-E-B

b Decide if the following walks are circuits. Answer Yes or No.
 i A-B-C-D-A ii B-C-D-C-B

c Decide if the following walks are paths. Answer Yes or No.
 i C-E-B-C-D ii E-B-C-D-A

d Decide if the following walks are cycles. Answer Yes or No.
 i C-D-A-B-C ii E-B-A-D-C-B-E

SOLUTION	EXPLANATION
a i Yes	Edges are only used once.
ii Yes	Edges are only used once.
b i Yes	Edges are only used once and it starts and ends at the same vertex.
ii No	The edge BC and CD are used twice.
c i No	The vertex C is visited twice.
ii Yes	Edges are used only once and vertices are visited once.
d i Yes	The cycle is a path that starts and ends at the same vertex.
ii No	The walk visits vertex B twice and uses the edge BE twice.

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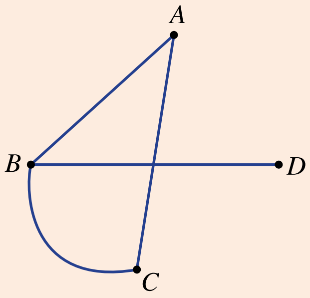
Now you try

Consider this graph.



a Decide if the following walks are trails. Answer Yes or No.
  i *D-B-A-C-A* **ii** *C-B-D*

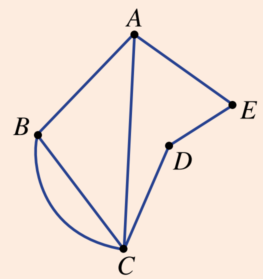
b Decide if the following walks are circuits. Answer Yes or No.
  i *D-B-C-B-D* **ii** *B-C-A-B*

c Decide if the following walks are paths. Answer Yes or No.
  i *D-B-A-C* **ii** *C-B-A-C-B*

d Decide if the following walks are cycles. Answer Yes or No.
  i *D-B-C-B* **ii** *B-C-A-B*

Example 7 Exploring Eulerian trails

Decide if this graph has an Eulerian trail. If so decide if all such trails will be circuits.



SOLUTION

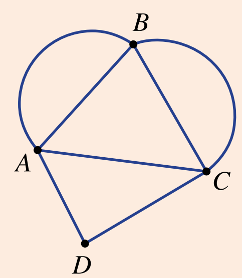
Yes e.g. *A-E-D-C-A-B-C-B*
Eulerian trails will not be circuits.

EXPLANATION

There are two vertices of odd degree and therefore an Eulerian trail exists.
For an Eulerian circuit to exist there must be zero vertices of odd degree.

Now you try

Decide if this graph has an Eulerian trail. If so decide if all such trails will be circuits.



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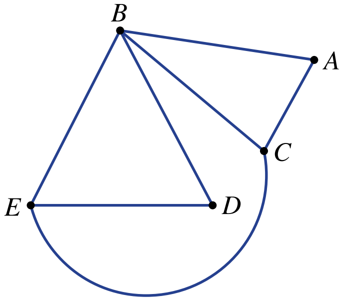
Exercise 11C

FLUENCY	1, 2, 4	1, 3–5	3–5

Example 6

1 Consider this graph.



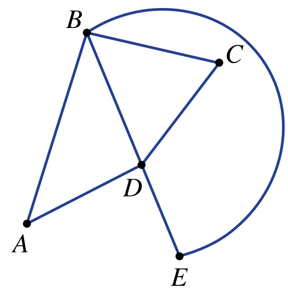
a Decide if the following walks are trails. Answer Yes or No.
  i A-B-D-E-B **ii** E-C-A-B-E-D

b Decide if the following walks are circuits. Answer Yes or No.
  i B-A-C-B-D-E-B **ii** D-E-C-B-E-D

c Decide if the following walks are paths. Answer Yes or No.
  i A-B-E-D-B-C **ii** A-C-B-E-D

d Decide if the following walks are cycles. Answer Yes or No.
  i D-E-B-D **ii** C-A-B-E-D-B-C

2 Consider this graph.



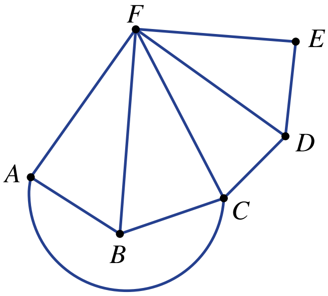
a Decide if the following walks are trails. Answer Yes or No.
  i A-B-C-D-B-E **ii** A-D-C-B-A-D

b Decide if the following walks are circuits. Answer Yes or No.
  i A-D-E-B-D-C-B-A **ii** D-C-B-A-D-E

c Decide if the following walks are paths. Answer Yes or No.
  i D-E-B-D-C **ii** C-D-E-B-A

d Decide if the following walks are cycles. Answer Yes or No.
  i D-E-B-C-D **ii** A-D-E-B-D-A

3 Consider this graph.



a Decide if the following walks are trails. Answer Yes or No.
  i A-B-C-A-F-D **ii** D-C-A-F-C-D

b Decide if the following walks are circuits. Answer Yes or No.
  i F-D-C-A-B-C-F **ii** B-F-C-A-F-B

c Decide if the following walks are paths. Answer Yes or No.
  i A-B-C-D-E **ii** F-C-B-F-A

d Decide if the following walks are cycles. Answer Yes or No.
  i D-F-A-C-D **ii** C-F-B-A-F-C

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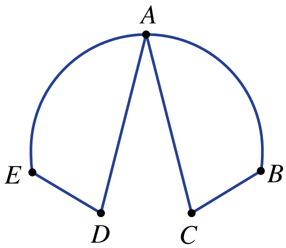
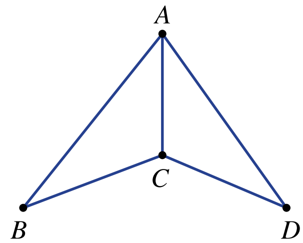
Example 7

4 Decide if the following graphs have an Eulerian trail. If so decide if all such trails will be circuits.

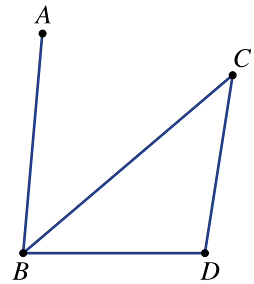
a



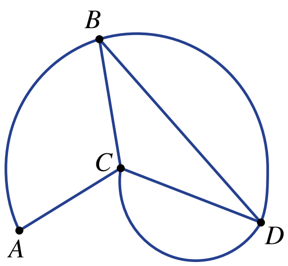
b



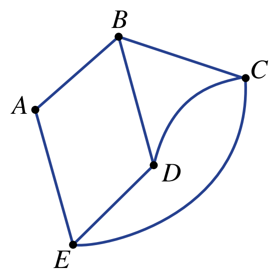
c



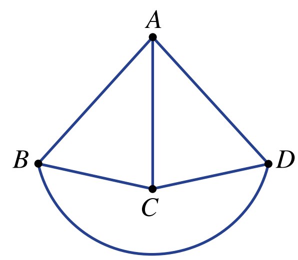
d



e

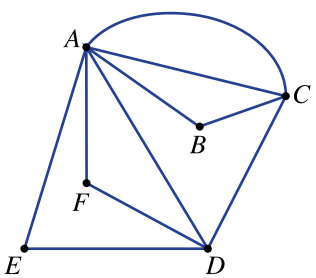


f

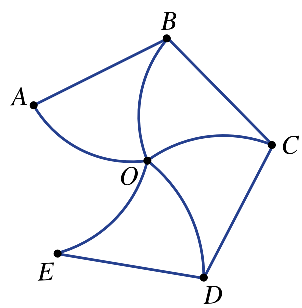


5 By determining the degree of each vertex decide if the following graphs have Eulerian trails.

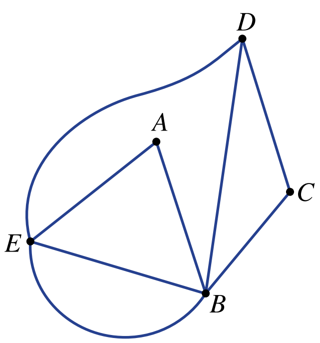
a



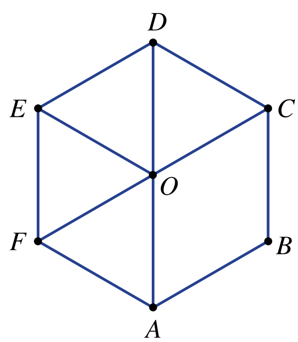
b



c



d



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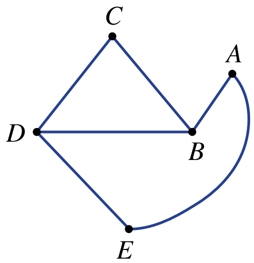
PROBLEM–SOLVING	6, 7	6–8	7–9

6 Consider the possible Eulerian trails in this graph.

a List the different Eulerian trails starting at vertex B and ending at vertex D.

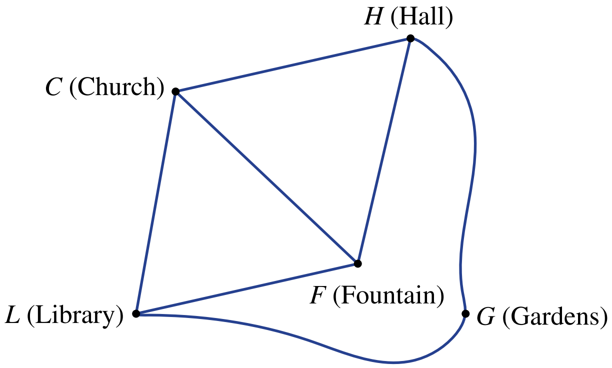
b List the different Eulerian trails starting at vertex D and ending at vertex B.



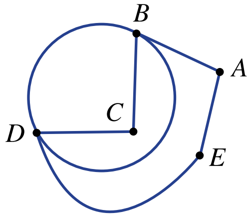
7 Margo wishes to visit every attraction in a village which are connected with walking paths as shown.

a Is it possible for Margo to find a walk which visits all the attractions starting and ending at the church without passing any given attraction twice? If so, how many ways can this be achieved?

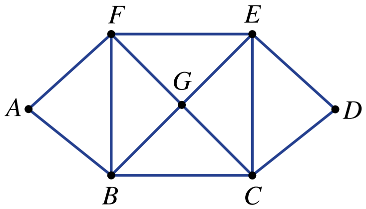
b Is it possible for Margo to use every footpath exactly once and visit every attraction at least once? If so, list the trail.



8 How many Eulerian circuits can you find through this network starting at vertex A? Do not count the same circuit in reverse.



9 Does this graph have an Eulerian circuit? If so find one.



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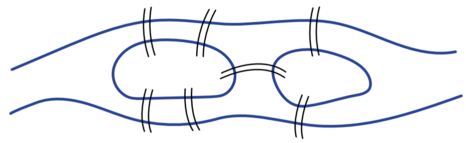
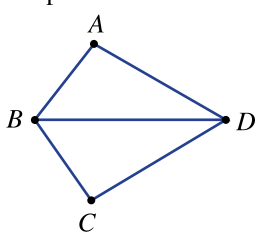
REASONING	10	10, 11	10–12

10 Let's revisit the Konigsberg bridge problem using this diagram.

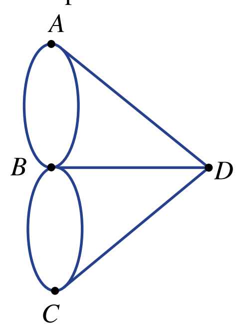


a Which of the Graphs 1 to 4 below is valid for the Konigsberg bridge problem?
b What do the edges on the network represent in the Konigsberg bridge problem?
c How many vertices on the graph representing the problem are of odd degree?
d What does your answer to part **c** tell you about the Konigsberg bridge problem?

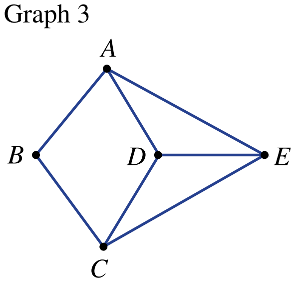
Graph 1



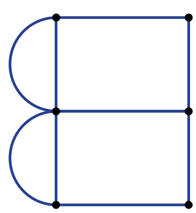
Graph 2

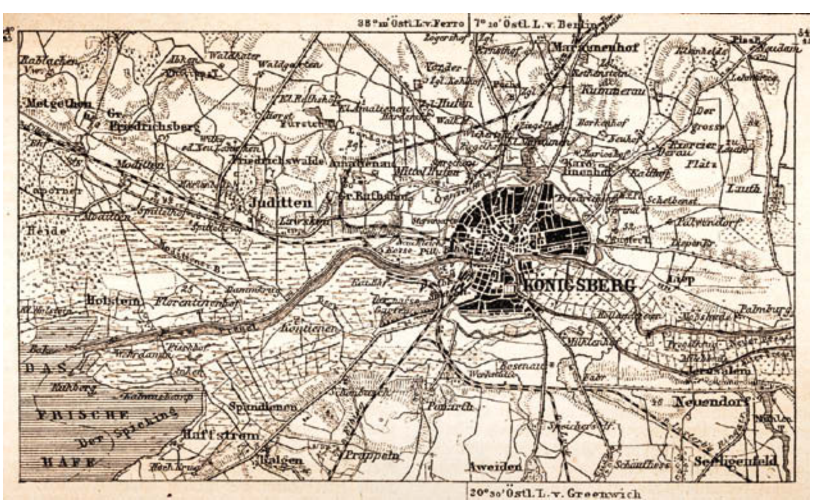


Graph 3



Graph 4





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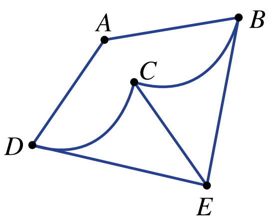
11 Answer true (T) or false (F).

a All paths are walks.
b All cycles are paths.
c All walks are trails.
d All trails are circuits.
e All paths are trails.
f All circuits are cycles.

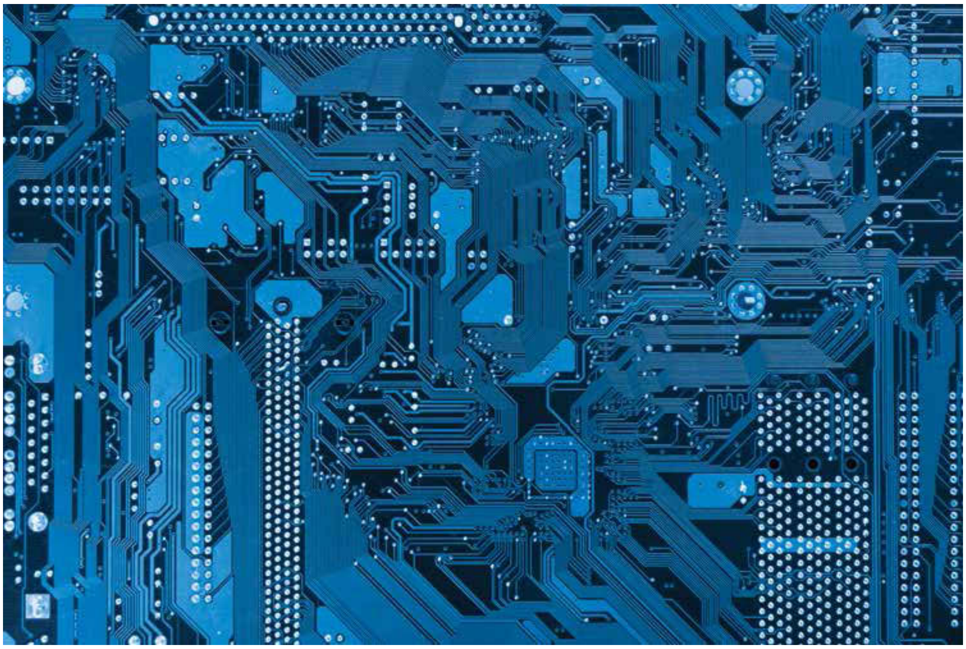
12 Semi-Eulerian graphs have an Eulerian trail but no Eulerian circuit. What can be said about the degree of the vertices in such a network? Try drawing an example.

ENRICHMENT: Adding edges to form Eulerian trails and circuits	–	–	13

13 Consider the given graph.



a By considering the degree of each vertex decide if the graph has the following:
  i an Eulerian trail
  ii an Eulerian circuit
b Is it possible to add a single edge so that the graph will have an Eulerian trail? If so, give an example.
c Is it possible to add a single edge so that the graph will have an Eulerian circuit? If so, give an example.
d What is the minimum number of edges that need to be added so that the graph has an Eulerian circuit?



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11D Shortest path problems

Learning intentions for this section:

- To understanded what a weighted graph is
- To interpret weighted graphs
- To be able to find a shortest path through a network

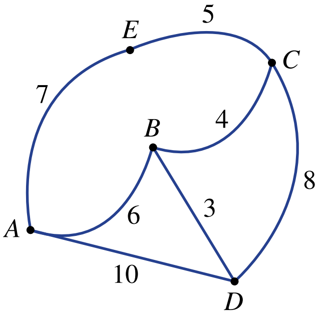
Past, present and future learning:

- This section has virtually no prerequisite knowledge
- It addresses a Path Topic for students progressing to Standard
- It has no relevance to Stage 6 Advanced
- It will be revised and extended in Year 12 Standard

A weighted graph is a network where each edge is labelled with a number. These numbers could represent the cost to transport goods between points, the voltages in an electrical circuit or distances between towns on a map. In this section we will focus on networks including distances, namely, shortest path problems. It is common in network theory to try to minimise the distances between two points. An example is a bus network where we might be interested in stopping at a range of points across a town using the minimum possible distance.

Lesson starter: Village distances

This simplified map shows the distance, in kilometres, between five villages Almore (A), Bellan (B), Coldstom (C), Denont (D) and Elimono (E).



- Find the total distance between Denont and Elimono if travelling via:
  - Bellan and Coldstom
  - Bellan and Almore.
- Find the shortest path from Denont to Elimono and state this minimum distance.
- If the distance from Denont to Bellan was instead 4 km, would this change your mind when finding the minimum distance between villages Denont and Elimono? Discuss.

KEY IDEAS

- A **weighted graph** is a graph with numbers attached to each of the edges.
  - These numbers could represent for example, costs, volumes, times or distances.
- A shortest path problem involves finding a walk through a network which provides a minimum distance.

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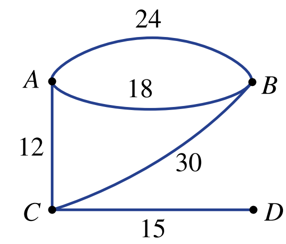
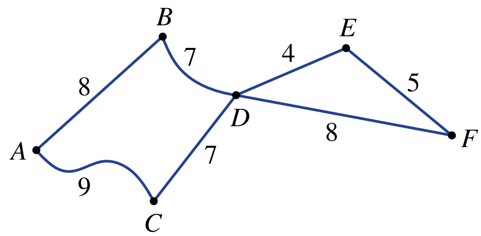
BUILDING UNDERSTANDING

1 This graph represents a rail map joining four towns A, B, C and D. Distances are in kilometres.



- **a** How many edges are labelled representing distances greater than 20 km?
- **b** How many different ways could you travel from town A to town C without using an edge more than once?
- **c** How many different ways could you travel from town A to town C without visiting a town more than once?
- **d** What is the length of the shortest path between towns A and C?
- **e** What is the length of the shortest path between towns A and D?

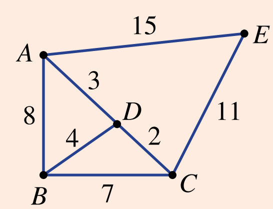
2 This graph represents the distance in kilometres using trails between points on a bushwalking map.



- **a** Find the distance between points A and D via:
  - **i** B
  - **ii** C.
- **b** Find the distance between points D and F via point E.
- **c** What points would be visited if the shortest path is taken from point A to point F?

Example 8 Interpreting a weighted graph

This graph shows the distance, in metres, between vertices in a pipe network.

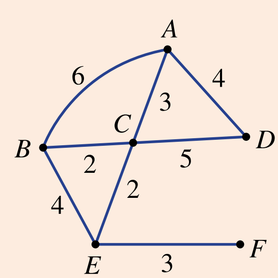


- **a** How far is it from point B to C via D on the network?
- **b** Calculate the distance for the path A-D-B-C-E.

SOLUTION	EXPLANATION
a $4 + 2 = 6$ m	The distance from B to D is 4 m and from D to C is 2 m.
b $3 + 4 + 7 + 11 = 25$ m	Add all the distances connecting the points in the correct order.

Now you try

This graph shows the distance, in metres, between vertices on a electrical circuit.



- **a** How far is it from point A to F via only B and E on the circuit?
- **b** Calculate the distance for the path A-D-C-E-F.

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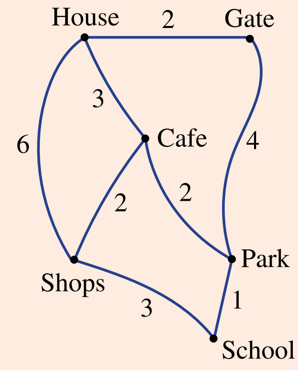
Example 9 Finding the shortest path

This graph represents a cycling network between a student's house and their school. Distances are in kilometres.



a How far is the walk from the house to the school via:
- **i** the shops without passing by the café?
- **ii** the gate and the park without passing by the café?

b Find the minimum distance from the house to the school via:
- **i** the shops
- **ii** any possible walk.

SOLUTION

a **i** 6 + 3 = 9 km

ii 2 + 4 + 1 = 7 km

b **i** 3 + 2 + 3 = 8 km

ii 3 + 2 + 1 = 6 km

EXPLANATION

The path House-Shops-School includes the distances 6 km and 3 km.

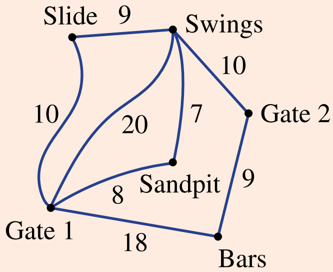
The path House-Gate-Park-School includes the distances 2 km, 4 km and 1 km.

It is shorter to get to the shops via the café.

Some possible walks are:
- House-Gate-Park-School (7 km)
- House-Shops-School (9 km)
- House-Café-Shops-School (8 km)
- House-Café-Park-School (6 km)

Now you try

This graph represents a path network connecting equipment in a playground. Distances are in metres.



a How far is the path from gate 1 to the swings via:
- **i** the slide?
- **ii** the sandpit?

b Find the minimum distance from gate 1 to gate 2 via:
- **i** the slide
- **ii** any possible walk.

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Exercise 11D

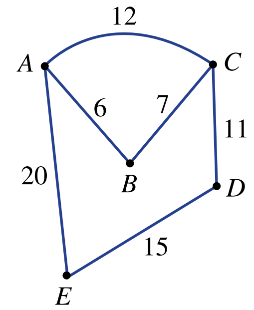
FLUENCY	1–4	1, 3, 4	3, 4

Example 8

1 This graph shows the distance, in kilometres, for a simple road network between points A, B, C, D and E

a How far is it from point E to C via D on the network?

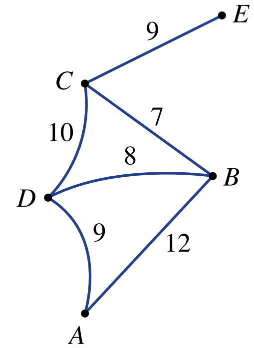
b Calculate the distance for the path A-B-C-D.



2 This graph shows the distance, in metres, between junctions on a cable network.

a How far is it from point A to B via D and C on the network?

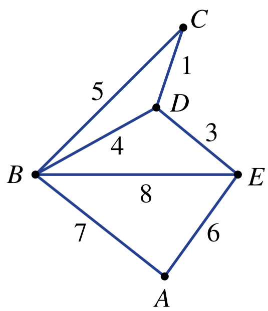
b Calculate the distance for the path A-B-D-C-E.



3 This graph shows the distance, in centimetres, between points on an electrical circuit.

a How far is it from point A to B via E, D and C on the network?

b Calculate the distance for the path A-E-B-C.



Example 9

4 This graph represents a street map connecting a number of key places in a town. Distances are in kilometres.

a How far is the walk from the post office to the park via:

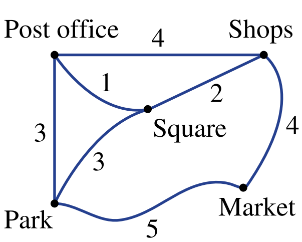
i the square without passing by the shops?

ii the square, shops and market?

b Find the minimum distance from the shops to the park via:

i the post office

ii any possible walk.



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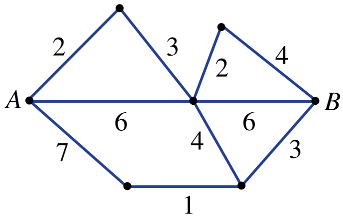
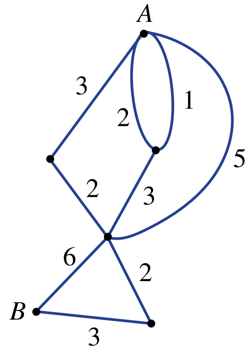
PROBLEM–SOLVING	5, 6	5, 6	5(½), 6, 7

5 Find the shortest possible distance when travelling from point A to point B in these graphs. Distances are in kilometres.

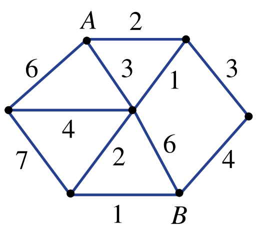
a



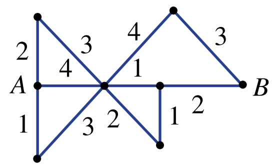
b



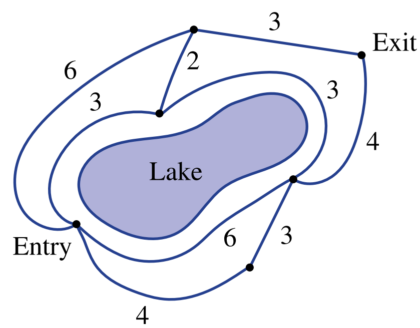
c



d

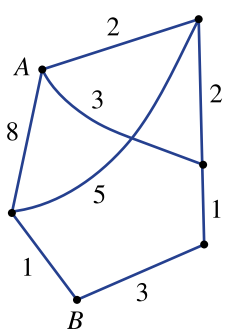


6 A network of walking paths around a lake are represented in this graph with distances in kilometres. Find the shortest distance between the entry and exit points.

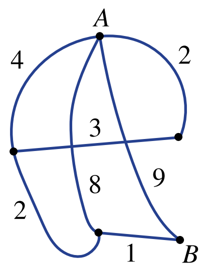


7 These graphs have intersecting edges but are in fact planar. Find the shortest distance from point A to point B. Distances are in centimetres.

a



b



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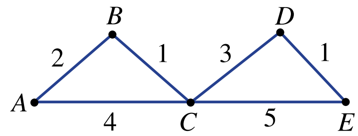
REASONING	8	8, 9	8–10

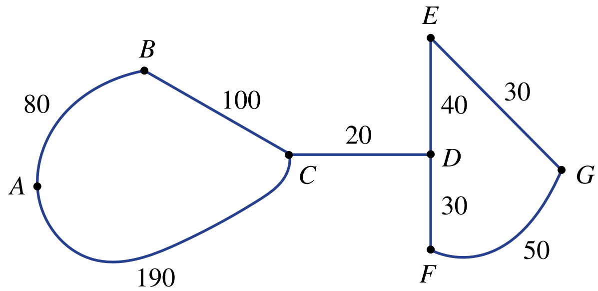
8 Explain why the shortest distance between the points A and E in this graph involves going via points B and D rather than just via point C.



9 This weighted graph represents the cost of transporting goods via a network. The units are in dollars.

a Which points should be visited to transport goods with minimum cost from point A to point G?

b What is the minimum cost if transporting the goods from point A to point G?



10 An Eulerian trail follows every edge of a graph with no repeated edges. Consider this graph.

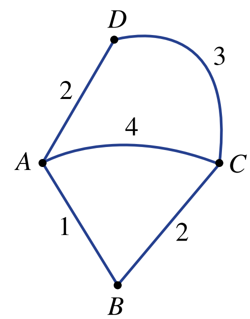
a Decide if the following are Eulerian trails.

i A-B-C-D

ii A-D-C-A-B

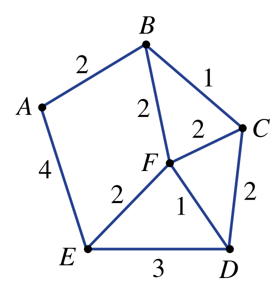
iii A-B-C-A-D-C

b True or False? Eulerian trails will have the same total distance.



ENRICHMENT: The travelling salesperson	–	–	11

11 A travelling salesperson problem involves completing a circuit of a network so that each vertex is visited exactly once using a minimum distance and returning to the starting point. This is also called a Hamiltonian circuit where the only vertex visited twice is the start and end point of the circuit. This weighted graph represents the distance between houses visited by a salesperson. Distances are in kilometres.



a Decide if the following walks are Hamiltonian circuits.

i A-B-E-A

ii A-B-C-D-E-A

iii A-B-F-D-E-A

iv A-E-F-D-C-B-A

b Find the distance of the Hamiltonian circuit A-B-F-C-D-E-A.

c Find the length of the minimum Hamiltonian circuit; that is, find the shortest distance a salesperson can travel if visiting each point exactly once and starting and ending at point A.

d If the salesperson started at a different point, is there a shorter total distance the salesperson can travel compared to your answer in part **c** above?

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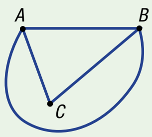
Chapter summary

Geometry and networks

Networks

A network is a diagram connecting vertices (nodes) using lines (edges)
the degree of a vertices is the number of edges connected to it



B is an odd vertex as it has an odd number of edges connected to it

Planar graphs and Euler's formula

Isomorphic graphs (same edges, vertices and connections) can be redrawn to look the same.
A planar graph can be drawn so it has no intersecting edges.
Euler's formula for planar graphs says:
$v + f = e + 2$ or alternatively $v - e + f = 2$
v = no. of vertices , f = no. of faces, e = no. of edges

Weighted graphs

A weighted graph has a number attached to each edge which might represent distances, time, costs etc.

A shortest path problem involves finding a walk through the network of minimum distance.

Walks

A walk is a sequence of edges though network:
- trail – walk with no repeated edges
- path – a walk with no repeated edges velices
- cycle – a path that starts and ends at same vertices
- circuit – a trail that starts and ends at same vertices
- Eulerian trail – every edge is used exactly once, needs 0 or 2 odd vertices to exist
- Eulerian circuit – an Eulerian trail that starts and ends at same vertex, requires no odd vertices.

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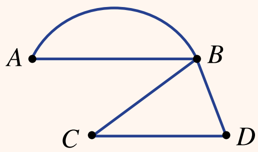
Chapter checklist with success criteria

A printable version of this checklist is available in the Interactive Textbook ✓

11A

1. I can find walks between vertices.
e.g. For the graph shown, find how many walks there are connecting *A* and *C* without visiting a vertex more than once or using an edge more than once.



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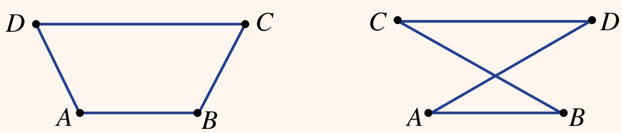
11B

2. I can decide if graphs are isomorphic.
e.g. Decide if the following two graphs are isomorphic.



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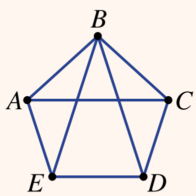
11B

3. I can decide if a graph is planar non-planar.
e.g. Decide if the following graph is planar non-planar.



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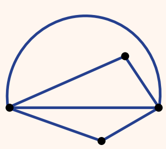
11B

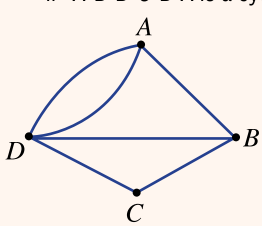
4. I can verify Euler's formula.
e.g. Find the number of vertices, edges and faces in this planar graph and verify Euler's formula.



☐

11C

5. I can define a type of walk.
e.g. For the given graph, decide if:
i *A*-*B*-*C*-*D*-*B* is a trail
ii *A*-*B*-*D*-*C*-*B*-*A* is a cycle.



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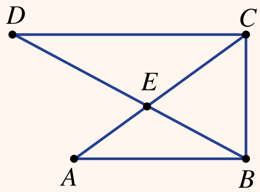
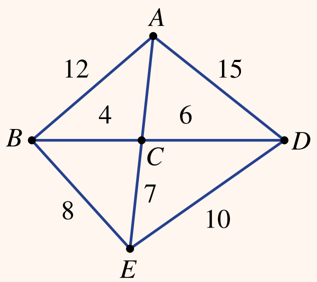
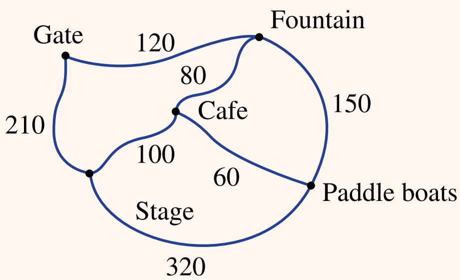
Chapter checklist

Chapter checklist with success criteria

		✓
11C	**6. I can decide if a graph has an Eulerian trail or circuit.** e.g. Decide if this graph has an Eulerian trail and if so will all such trails be circuits.  	☐
11D	**7. I can interpret a weighted graph.** e.g. This graph shows the distances in kilometres along the roads connecting five towns, *A* to *E*. Calculate the distance for the path *A*-*B*-*C*-*E*-*D*.  	☐
11D	**8. I can find the shortest path on a network graph.** e.g. This graph represents a path network connecting landmarks in the Botanical gardens. Distances are in metres. Find the minimum distance from the Gate to the Paddle boats.  	☐

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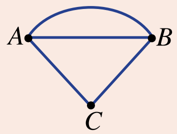
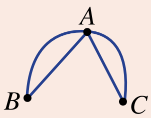
Short-answer questions

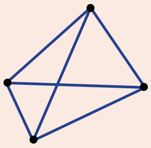
11B **1** **a** Decide if the following two graphs are isomorphic.



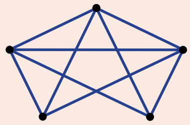


b Decide if the following graphs are planar non-planar.

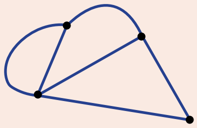
i



ii

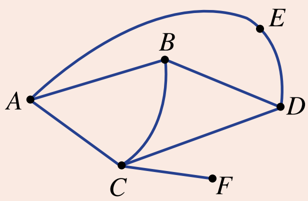


11A/B **2** Consider the planar graph shown.



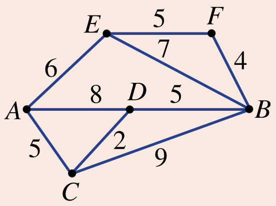
a State the number of edges and the number of vertices.
b State the number of faces.
c Hence, verify Euler's formula for the graph.

11C **3** Consider the graph shown.



a Decide if the walk A-C-D-F-C-A is a trail.
b Explain why this graph does not have an Eulerian trail.
c Add one edge to the graph from D so that it has an Eulerian trail.
d State the Eulerian trail from part **c** and decide if it is a circuit.

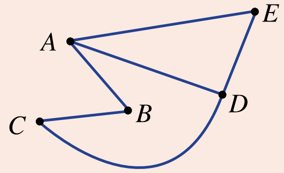
11D **4** This graph shows the distances between points on an electrical circuit in centimetres.



a Calculate the distance for the path A-E-F-B.
b What is the shortest distance from A to B?

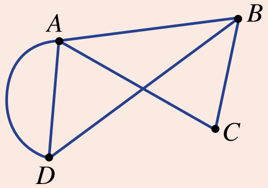
Multiple-choice questions

11A **1** How many odd vertices does the graph shown have?



A 0 **B** 1 **C** 2
D 3 **E** 4

11C **2** So that the graph shown has an Eulerian circuit, which edge should be removed?



A A-D **B** A-C **C** B-D
D A-B **E** B-C

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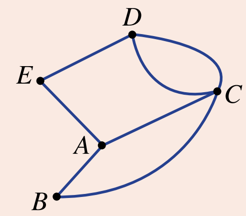
Chapter review

11C

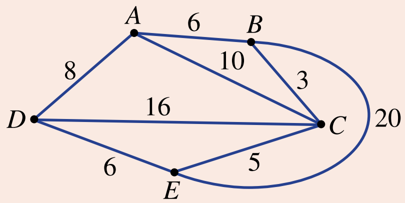
3 Which of the following is not a path for the graph shown?



A *A-B-C-D* **B** *A-C-D-E* **C** *E-A-B-C-D*
D *A-C-D-C-B* **E** *E-D-C-A-B*

11D

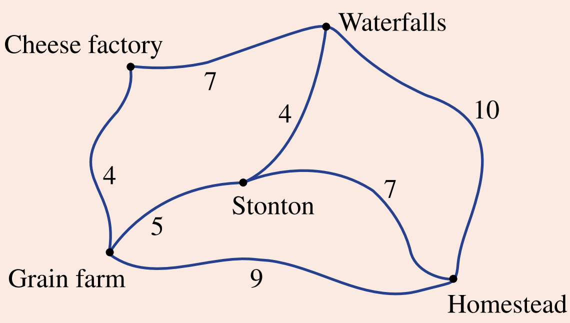
4 The shortest path through this network from A to D via B, where distance is in kilometres, is:



A 20 km **B** 25 km **C** 32 km
D 8 km **E** 19 km

Extended-response questions

1 This graph represents a map of the countryside near the village of Stonton. Distances are in kilometres.



a Find the sum of degrees of all the vertices.
b How far is the walk Stonton-Grain farm-Cheese factory?
c Confirm that the graph satisfies Euler's formula $v + f = e + 2$.
d Decide if the trail Stonton-Waterfalls-Homestead-Grain farm-Cheese factory is Eulerian. Give a reason.
e Find the length of the shortest path between the following places.
  i Grain farm and Waterfalls
  ii Homestead and Cheese factory if going via Stonton

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