UNSW SYDNEY

MATH5805 – A Modern Introduction to Extreme Value Theory

Week 9 - Multivariate Extremes: a Point Process Representation (I)

Dr Boris Béranger
School of Mathematics and Statistics
UNSW Sydney

CRICOS Provider Code 00098G

Lecture outline

1. Introduction & Recap
2. The Poisson Process Representation
3. Extremal Dependence Structure
4. Quantifying and Representing Extremal Dependence
5. Parametric Models for Extremal Dependence

1. Introduction & Recap



Goals

 This lecture is about introducing **a Poisson Point Process representation of multivariate extremes Extremes**. In particular, we

- Use a transformation to pseudo-polar coordinates
- Introduce an estimation procedure which relies on the angular density.

Note that throughout this lecture we will make the assumption that $\{\mathbf{X}_i = (X_{i,1}, \ldots, X_{i,d}), i = 1, \ldots, n\}$ is a sequence of i.i.d. d-dimensional random variables with **common joint distribution function** F and **marginal distributions** $F_1, \ldots, F_d$.

Recap

Let's summarise what we have learned about the **multivariate extremes** so far.

Recap of Week 7 & 8

- We define the **vector of componentwise maxima**

$$\boldsymbol{M_n} = (M_{n,1}, \ldots, M_{n,d}), \quad \text{where } M_{n,j} = \max_{i=1,\ldots,n} X_{i,j}.$$

- If $\exists \boldsymbol{a_n} = (a_{n,1}, \ldots, a_{n,d}) > \mathbf{0} = (0, \ldots, 0)$ and $\boldsymbol{b_n} = (b_{n,1}, \ldots, b_{n,d}) \in \mathbb{R}^d$ s.t

$$\mathbb{P}\left(\frac{\boldsymbol{M_n} - \boldsymbol{b_n}}{\boldsymbol{a_n}} \leq \mathbf{y}\right) = F^n(\boldsymbol{a_n}\mathbf{y} + \boldsymbol{b_n}) \xrightarrow{n\to\infty} G(\mathbf{x}), \tag{1}$$

for all continuity points $\mathbf{x}$ of a non-degenerate distribution G **(MEVDs)**.

Recap

Recap of Week 7 & 8 (continued)

- $M^*_{n,j} \xrightarrow{d} Y_j \sim G_{\xi_j}$ for $j = 1, \ldots, d$. The margins are GEV distributed!

- If $\tilde{Y}_j = (1 + \xi_j Y_j)_+^{-1/\xi_j}$ then it is unit Fréchet distributed.

- Consider $Z_{i,j}$ to be a transformation of $X_{i,j}$ with unit Fréchet marginal distributions. Let $\widetilde{M}_{n,j} = \max_{i=1,\ldots,n} Z_{i,j}/n, j = 1, \ldots, d$ then for $z_1, \ldots, z_d > 0$

$$\left(\widetilde{M}_{n,1} \leq z_1, \ldots, \widetilde{M}_{n,d} \leq z_d\right) \xrightarrow{d} \mathbf{Y} \sim G(z_1, \ldots, z_d) = \exp\left\{-\mathbf{V(z_1, \ldots, z_d)}\right\},$$

where G is a non-degenerate distribution function and V is the **exponent (dependence) function**.

Recap

The exponent function represents the dependence structure of multiple extremes (extremal dependence). It depends on the unknown F.

Theorem - Characterisation of the exponent function (Week 7 - slide 24)
The exponent function defined in the previous Theorem can be written as

$$V(z_1, \ldots, z_d) = d \int_{\mathbb{W}} \max\left(\frac{\omega_1}{z_1}, \ldots, \frac{\omega_d}{z_d}\right) \mathrm{d}H(\boldsymbol{\omega}),$$

where $\mathbb{W} = \{\boldsymbol{\omega} \in \mathbb{R}_+^d \backslash \{\mathbf{0}\} : \omega_1 + \cdots + \omega_d = 1\}$, i.e. the d-dimensional unit simplex and $H(\cdot)$ is the **spectral measure** satisfying

$$\int_{\mathbb{W}} \omega_j \mathrm{d}H(\boldsymbol{\omega}) = \frac{1}{d}, \quad \forall j = 1, \ldots, d.$$

If H is differentiable with density h known as the *angular density*, we can write $\mathrm{d}H(\boldsymbol{\omega}) = h(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}$. **Details about this results will unravel throughout this lecture.**

Example - Independence

If H is not differentiable but places mass $1/d$ on the d-dimensional vectors $\boldsymbol{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, then

$$V(z_1, \ldots, z_d) = -\frac{1}{z_1} - \ldots - \frac{1}{z_d} = -\sum_{j=1}^{d} \frac{1}{z_j},$$

and thus

$$G(z_1, \ldots, z_d) = \prod_{j=1}^{d} \exp\left\{-\frac{1}{z_j}\right\} = \prod_{j=1}^{d} G(\infty, \ldots, z_j, \ldots, \infty).$$

Since the joint distribution is the product of the marginal distributions we are in presence of independent random variables.

Example - Perfect dependence

If H is not differentiable but places mass 1 on $\boldsymbol{\omega} = \left(\frac{1}{d}, \ldots, \frac{1}{d}\right)$, then

$$V(z_1, \ldots, z_d) = \max\left(\frac{1}{dz_1}, \ldots, \frac{1}{dz_d}\right),$$

which implies that

$$G(z_1, \ldots, z_d) = \exp\left\{-\max\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right)\right\}$$

The above is the joint distribution of variables that are marginally standard Fréchet, but which are perfectly dependent: $X_1 = \ldots = X_d$ with probability 1.

2. The Poisson Process Representation



Poisson Point Process representation

We start by providing some results about limiting results of componentwise maxima.

Theorem [dHF06, Theorem 6.1.]

There is equivalence between

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{i=1,\dots,n} Z_{i,1}/n \le z_1, \dots, \max_{i=1,\dots,n} Z_{i,d}/n \le z_d\right) = G(z_1,\dots,z_d),$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{i=1,\dots,n} X_{i,1} \le U_1(nz_1), \dots, \max_{i=1,\dots,n} X_{i,d} \le U_d(nz_d)\right) = G(z_1,\dots,z_d),$$

where, as in Week 3, $U_i(x) = F_i^{\leftarrow}(1-1/x), i = 1,\dots,d$.

Poisson Point Process representation

From the theorem we can also write:

$$\lim_{n\to\infty} n\left[1 - F(U_1(nz_1), \ldots, U_d(nz_d))\right] = -\log G(z_1, \ldots, z_d).$$

Construction

Construct a sequence of point processes on $\mathbb{R}^{d+1}$ by

$$N_n = \left\{ \left(\frac{i}{n+1}, \frac{Z_{i,1}}{n}, \ldots, \frac{Z_{i,d}}{n} \right) : i = 1, \ldots, n \right\} \tag{2}$$

For some region $A = [t_1, t_2] \times A_{\mathbf{z}}$ where $A_{\mathbf{z}} = \mathbb{R}^d_+ \setminus \{(\mathbf{0}, \mathbf{z})\}$ and $\mathbf{z} > \mathbf{0}$, the intensity measure can be written as

$$\begin{aligned} \Lambda_n(A) &= (t_2 - t_1) \times n\left[1 - F\left(U_1(nz_1), \ldots, U_d(nz_d)\right)\right] \\ &\equiv (t_2 - t_1) \times \eta_{\mathbf{n}}\left(\mathbf{A_z}\right) \end{aligned}$$

Some justification

Using the previous theorem, the probability of each point in N_n falling into $A = [0,1] \times A_{\mathbf{z}}$ is

$$\begin{aligned} p_n &= 1 - \mathbb{P}\left(\frac{Z_{i,1}}{n} \leq z_1, \ldots, \frac{Z_{i,d}}{n} \leq z_d\right) \\ &\approx 1 - \mathbb{P}\left(X_{i,1} \leq U_1(nz_1), \ldots, X_{i,d} \leq U_d(nz_d)\right). \end{aligned}$$

Poisson Point Process representation

Theorem [dHF06, Theorem 6.1.5, 6.1.11]

If the limiting results such as those in the previous theorem hold, then

$$N_n \xrightarrow{d} N, \quad \text{as } n \to \infty,$$

where N **is a non-homogeneous Poisson point process** with intensity measure on A

$$\begin{aligned}\Lambda(A) &= (t_2 - t_1)\,\eta\,(A_{\mathbf{z}}) \\ &= (t_2 - t_1)\,\eta\left\{\left(\mathbf{v} \in \mathbb{R}_+^{\mathbf{d}} : \mathbf{v_1} > \mathbf{z_1} \text{ or } \ldots \text{ or } \mathbf{v_d} > \mathbf{z_d}\right)\right\} \\ &= (t_2 - t_1)\mathbf{V}(\mathbf{z_1}, \ldots, \mathbf{z_d}).\end{aligned}$$

It is named the **exponent measure**.

Some further justification

Let $N_n(A)$ be the number of points of N_n in region A. Because the $\mathbf{X}_i$'s are i.i.d., $N_n(A)$ follows a binomial distribution $\text{Bin}(n, p_n)$ with $p_n \approx -\frac{1}{n} \log G(z_1, \ldots, z_d)$.

As $n \to \infty$, the binomial distribution converges to a Poisson distribution with rate

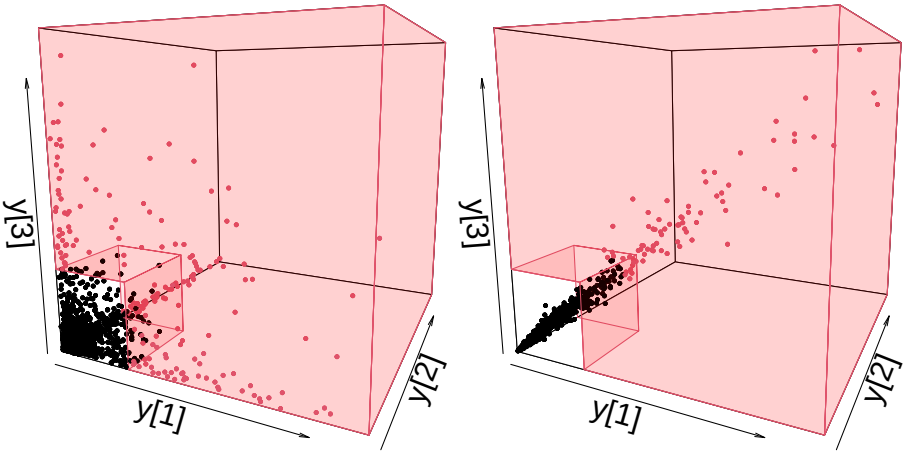
$$\Lambda(A) = -\log G(z_1, \ldots, z_d) = V(z_1, \ldots, z_d).$$

Interpretation

Numbering the rescaled observations that fall in a critical region where at least one coordinate is large, makes it possible for the limiting distribution to be computed using the void probability of N, that is

$$G(\mathbf{z}) = \mathbb{P}\left(N\left([0,1] \times A_{\mathbf{z}}\right) = 0\right) = \exp\left(-V(\mathbf{z})\right), \mathbf{z} > \mathbf{0}.$$

Poisson Point Process representation



Weak dependence (left): one coordinate is large. Strong dependence (right): all coordinates are large.

3. Extremal Dependence Structure



Extremal dependence structure

So far we have:

- Transformed the margins so that they are all equal (unit-Fréchet).
- Represented the standardised distribution using a Poisson Point Process.

Now we will:

- Specify the structure of the exponent measure.

To complete this task we will make a change to **pseudo-polar coordinates**.

Properties of the Poisson Point Process

With unit-Fréchet margins, the *max-stability* property of MEVDs can be re-written

$$G^a(a\mathbf{x}) = G(\mathbf{x}), \quad \text{for any } a > 0.$$

A direct implication is that the intensity measure of the Poisson Point Process is homogeneous of order $1-$, i.e.,

$$\Lambda(aA_{\mathbf{z}}) = a^{-1}\Lambda(A_{\mathbf{z}})$$

for all $A = [0,1] \times A_{\mathbf{z}}$ with $A_{\mathbf{z}} := \mathbb{R}_+^d \backslash (\mathbf{0}, \mathbf{z}] \subset \mathbb{R}_+^d$, $\mathbf{z} > \mathbf{0}$.

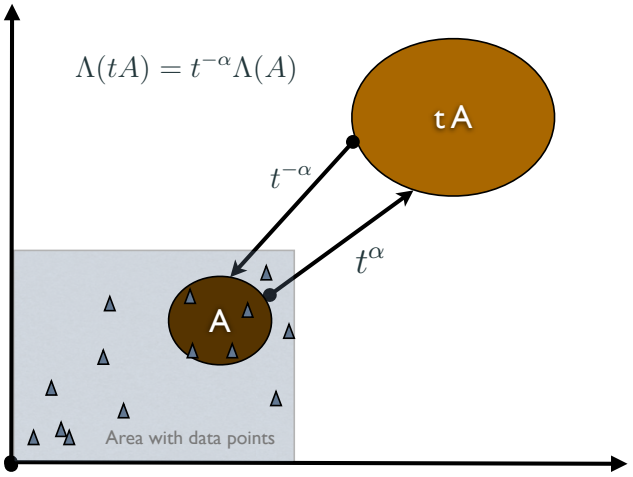
Note that for a Borel set $A \subset \mathbb{R}_+^d$ we have $aA = \{a\mathbf{v} : \mathbf{v} \in A\}$ and $A_{a\mathbf{z}} = aA_{\mathbf{z}}$.

Indeed we have,

$$G^a(a\mathbf{x}) = \exp\left(-\Lambda(a\mathbf{x})\right)^a = \exp\left(-a\Lambda(a\mathbf{x})\right)$$

Properties of the Poisson Point Process

The **homogeneity property** is essential for inference.



Spectral decomposition

Recall that the d-dimensional unit simplex is defined by

$$\mathbb{W} = \{\boldsymbol{\omega} \in \mathbb{R}_+^d \backslash \{\mathbf{0}\} : \omega_1 + \cdots + \omega_d = 1\},$$

and define the distance from $\mathbf{z} \in \mathbb{R}_+^d$ using the **sum-norm** by $\|\mathbf{z}\| = |z_1| + \cdots + |z_d|$.

Pseudo-polar coordinate transformation

Consider the transformation $Q : \mathbb{R}_+^d \backslash \{\mathbf{0}\} \longrightarrow (0, \infty) \times \mathbb{W}$, given by

$$(r, \boldsymbol{\omega}) := Q(\mathbf{z}) = \left(\|\mathbf{z}\|, \frac{\mathbf{z}}{\|\mathbf{z}\|} \right), \quad \mathbf{z} \in \mathbb{R}_+^d \backslash \{\mathbf{0}\},$$

where r is the radius and $\boldsymbol{\omega}$ the angle

Spectral decomposition

We want to characterise the **induced measure** $\psi(\mathcal{W}_r) = \eta(Q^{\leftarrow}(\mathcal{W}_r))$ for all sets $\mathcal{W}_r = r \times \mathcal{W}$ with $r > 0$ and $\mathcal{W} \subset \mathbb{W}$.

$$\begin{aligned}
\psi(\mathcal{W}_r) &= \eta\left\{\left(\mathbf{v} \in \mathbb{R}_+^d \backslash \{\mathbf{0}\} : \|\mathbf{v}\| > r, \frac{\mathbf{v}}{\|\mathbf{v}\|} \in \mathcal{W}\right)\right\} \\
&= \eta\left\{\left(r\mathbf{u} \in \mathbb{R}_+^d \backslash \{\mathbf{0}\} : \|\mathbf{u}\| > 1, \frac{\mathbf{u}}{\|\mathbf{u}\|} \in \mathcal{W}\right)\right\} \\
&= r^{-1} H'(\mathcal{W}) \qquad \text{(by homogeneity property)}
\end{aligned}$$



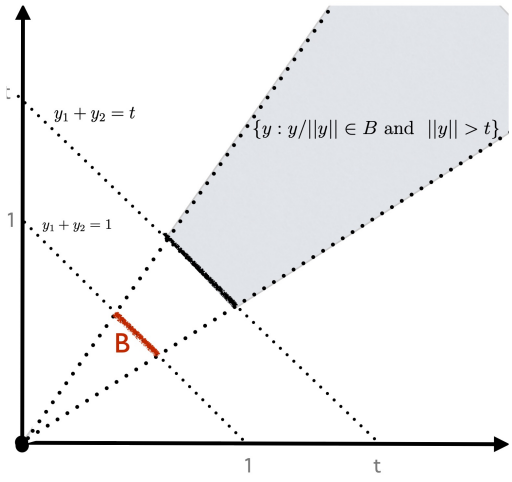
So why transforming to pseudo-polar coordinates?

The measure η becomes the product of two independent measures:

- **radial measure** $(1/r)$ quantifying the intensity from the origin
- **spectral (angular) measure** (H') quantifying the angular spread/direction.

Spectral decomposition

2-dimensional illustration



Here B is in the bivariate simplex, i.e.:

$$B \subset \{(y_1, y_2) \in \mathbb{R}^2_+ \backslash (0,0) : y_1 + y_2 = 1\}.$$

The grey shaded region corresponds to the points with angular components defined by B and radius $r = \|y\|$ greater than t.

Spectral decomposition

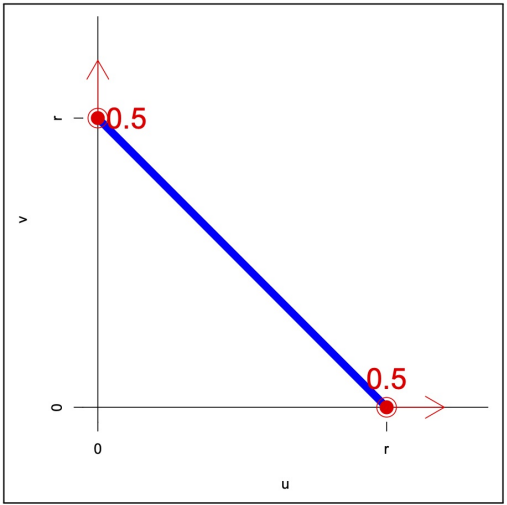
For $r > 0$ and $\boldsymbol{\omega} \in \mathbb{W}$, the **density** of ψ is $\mathrm{d}\psi(r, \boldsymbol{\omega}) = r^{-2}\mathrm{d}r \times \mathrm{d}H'(\boldsymbol{\omega})$.

For $A = [0,1] \times A_{\mathbf{z}}$, we have

$$\begin{aligned}
\Lambda(A) = \eta(A_{\mathbf{z}}) &= \psi\left\{Q(\mathbf{v} \in \mathbb{R}_+^d \backslash \{\mathbf{0}\}) : v_1 > z_1 \text{ or } \ldots \text{ or } v_d > z_d\right\} \\
&= \psi\left\{(r, \boldsymbol{\omega}) \in (0,\infty) \times \mathbb{W} : r > \min_{j=1,\ldots,d}\left(\frac{z_j}{\omega_j}\right)\right\} \\
&= \int_{\mathbb{W}} \int_{\min_{j=1,\ldots,d}\left(\frac{z_j}{\omega_j}\right)}^{\infty} \frac{1}{r^2}\mathrm{d}r \times \mathrm{d}H'(\boldsymbol{\omega}) \\
&= \int_{\mathbb{W}} \max_{j=1,\ldots,d}\left(\frac{\omega_j}{z_j}\right) \mathrm{d}H'(\boldsymbol{\omega}).
\end{aligned}$$

In pseudo-polar coordinates, extremes are the values whose radial component is higher than a high threshold.

Spectral decomposition



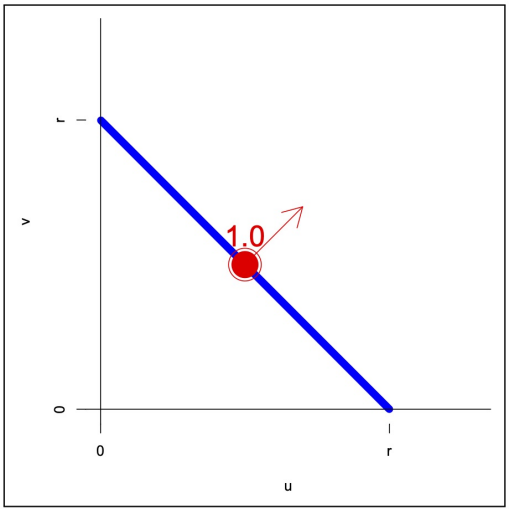
Example - Independence

$d = 2$. The **blue line** represents $\mathcal{W}_r = r \times \mathcal{W}$ with $\mathcal{W} = \mathbb{W}$, i.e. the line such that the radius $u + v$ is r.

The angles are either $\omega = (1, 0)$ or $(0, 1)$. The angular measure has density which puts point mass of $1/2$ at 0 and 1.

The interest is in the points lying on the sets $\{(u, 0); u \geq r\}$ and $\{(0, v); v \geq r\}$.

Spectral decomposition



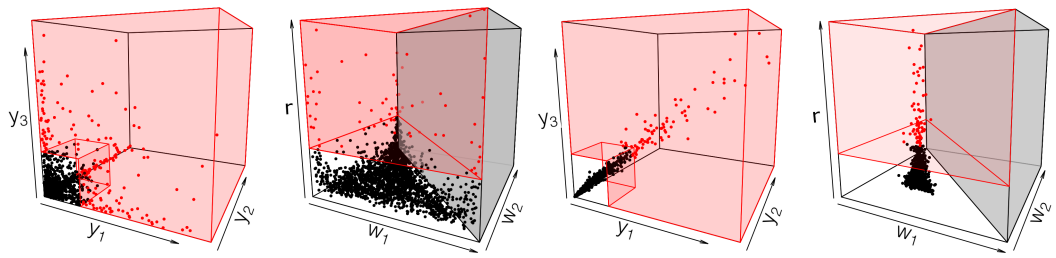
Example - Complete dependence

$d = 2$. The **blue line** represents $\mathcal{W}_r = r \times \mathcal{W}$ with $\mathcal{W} = \mathbb{W}$, i.e. the line such that the radius $u + v$ is r.

In the case of complete dependence there is only one angle $\boldsymbol{\omega} = (1/2, 1/2)$. The angular measure has density which puts point mass of 1 at $(1/2, 1/2)$.

The interest is in the points lying on the set $\{(u, u); u > r/2\}$.

Spectral decomposition



Weak dependence (left): one coordinate is large, angular coordinates around the vertices of the simplex. Strong dependence (right): all coordinates are large, angular components in the centre of the simplex.

Spectral decomposition

The measure H' can be any finite measure on $\mathbb{W}$ satisfying the first moment conditions:

$$\int_{\mathbb{W}} \omega_{\mathbf{j}} \mathrm{d}\mathbf{H}'(\boldsymbol{\omega}) = \mathbf{1}, \quad \forall \mathbf{j} = \mathbf{1}, \ldots, \mathbf{d}.$$

This guarantees that the **marginal distributions of G are unit Fréchet**.



Can you prove this statement?

The total mass of H' is then

$$H'(\mathbb{W}) = \int_{\mathbb{W}} (\omega_1 + \cdots + \omega_d) \mathrm{d}H'(\boldsymbol{\omega}) = \sum_{j=1}^{d} \int_{\mathbb{W}} \omega_j \mathrm{d}H'(\boldsymbol{\omega}) = d.$$

Spectral decomposition

We define the **probability measure** $H := H'/H'(\mathbb{W})$, satisfying

$$\int_{\mathbb{W}} \omega_j \mathrm{d}H(\boldsymbol{\omega}) = \frac{1}{d}, \quad \forall j = 1, \ldots, d.$$

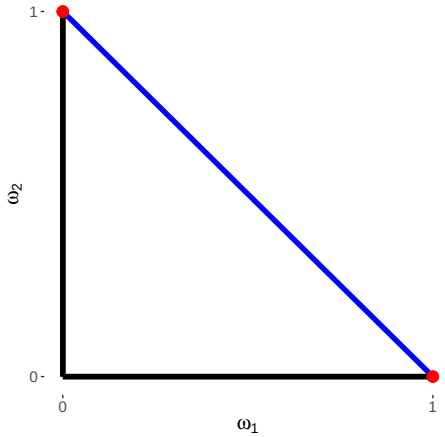
The MEVD with unit Fréchet margins is equal to

$$G(\mathbf{z}) = \exp\left\{-d \int_{\mathbb{W}} \max_{j=1,\ldots,d} \left(\frac{\omega_j}{z_j}\right) \mathrm{d}H(\boldsymbol{\omega})\right\},$$

as introduced in slide 7.

The measure H can place mass on the interior as well as on other subspaces of the simplex, such as edges and vertices.

Angular densities



Example - 2-dimensional simplex

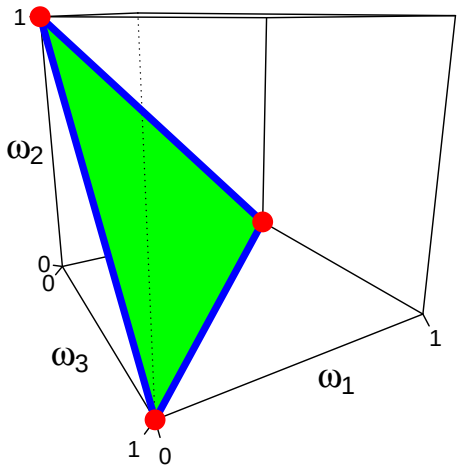
Here the simplex is represented by

$$\mathbb{W} = \{(\omega_1, \omega_2) > (0,0) : \omega_1 + \omega_2 = 1\}$$

"—" represents the **interior** of the simplex, i.e. any pairs (ω_1, ω_2) where $\omega_j \neq 0$ (or 1) for all $j = 1, 2$.

"•" represents the **vertices** of the simplex, i.e. the pairs $(\omega_1, \omega_2) = (1, 0)$ or $(0, 1)$.

Angular densities



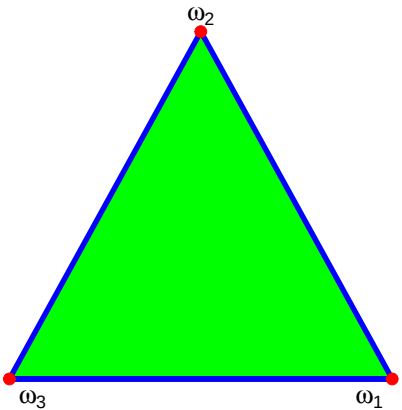
Example - 3-dimensional simplex

Here the simplex is represented by

$$\mathbb{W} = \{(\omega_1, \omega_2, \omega_3) > (0,0,0) : \omega_1 + \omega_2 + \omega_3\}$$

"▲" represents the **interior** of the simplex, i.e. any pairs $(\omega_1, \omega_2, \omega_3)$ where $\omega_j \neq 0$ (or 1) for all $j = 1, 2, 3$.

Angular densities



Example - 3-dimensional simplex

"—" represents the **edges** of the simplex, i.e. any pairs $(\omega_1, \omega_2, \omega_3)$ where **one** $\omega_j = 0$ for $j = 1, 2, 3$ and the other two components live in the 2-dimensional simplex.

"•" represents the **vertices** of the simplex, i.e. the pairs $(\omega_1, \omega_2, \omega_3) = (1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$.

Angular densities

Let $\mathbb{S}$ be the power set of $\{1, \ldots, d\}$ minus the empty set.

Let $\mathcal{S}$ be the index set that takes values in $\mathbb{S}$.

We define a subspace of the d-dimensional simplex $\mathbb{W}$ associated with $\mathcal{S}$ by

$$\mathbb{W}_{d,\mathcal{S}} = \{\boldsymbol{\omega} \in \mathbb{W} : \omega_j = 0 \text{ if } j \notin \mathcal{S}, \omega_j > 0 \text{ if } j \in \mathcal{S}\}$$

It provides a partition of $\mathbb{W}$ into $2^d - 1$ subsets.

Angular densities

Example - 2-dimensional simplex

Here $\mathbb{S} = \{\{1\}, \{2\}, \{1,2\}\}$. Then

- $\mathbb{W}_{2,\{1\}} = \{(1,0)\}$ and $\mathbb{W}_{2,\{2\}} = \{(0,1)\}$ represent the **vertices** of the simplex.
- $\mathbb{W}_{2,\{1,2\}} = \{(\omega, 1-\omega), \omega \in (0,1)\}$ represents the **interior** of the simplex.
- Number of subsets: $2^2 - 1 = 3$.

Angular densities

Example - 3-dimensional simplex

Here $\mathbb{S} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. Then

- $\mathbb{W}_{3,\{1\}} = \{(1,0,0)\}$, $\mathbb{W}_{3,\{2\}} = \{(0,1,0)\}$ and $\mathbb{W}_{3,\{3\}} = \{(0,0,1)\}$ represent the **vertices** of the simplex.
- $\mathbb{W}_{3,\{1,2\}} = \{(\omega, 1-\omega, 0), \omega \in (0,1)\}$, $\mathbb{W}_{3,\{1,3\}} = \{(0, \omega, 1-\omega), \omega \in (0,1)\}$ and $\mathbb{W}_{3,\{2,3\}} = \{(\omega, 0, 1-\omega), \omega \in (0,1)\}$ represent the **edges** of the simplex.
- $\mathbb{W}_{3,\{1,2,3\}} = \{(\omega_1, \omega_2, 1-\omega_1-\omega_2), \omega_1 \in (0,1), \omega_2 \in (0, 1-\omega_1)\}$ represent the **interior** of the simplex.
- Number of subsets: $2^3 - 1 = 7$.

Angular densities

We are now going to define the angular density h on all the subspaces of the simplex.

Denote by $h_{d,\mathcal{S}}$ **the density that lies on the subspace** $\mathbb{W}_{d,\mathcal{S}}$, $\mathcal{S} \in \mathbb{S}$.

Angular density on subspaces of the simplex

Let $\mathcal{S} = \{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}$, when G is absolutely continuous the angular density for any $\mathbf{z} \in \mathbb{R}_+^d$ is

$$\mathbf{h_{d,\mathcal{S}}}\left(\frac{\mathbf{z_{i_1}}}{\sum_{\mathbf{i}\in\mathcal{S}} \mathbf{z_i}}, \ldots, \frac{\mathbf{z_{i_{k-1}}}}{\sum_{\mathbf{i}\in\mathcal{S}} \mathbf{z_i}}\right) = -\left(\sum_{i\in\mathcal{S}} z_i\right)^{(k+1)} \lim_{\substack{z_j\to 0 \\ j\notin\mathcal{S}}} \frac{\partial^k V}{\partial z_{i_1} \cdots \partial z_{i_k}}(\mathbf{z}).$$

Comments

If $\mathbb{S} = \mathbf{e_i} = (0, \ldots, 0, 1, 0, \ldots, 0)$:

- $i = 1, \ldots, d$ then $\mathbb{W}_{d,\mathbf{e}_i}$ corresponds to the i-th vertex of the simplex.
- Since it corresponds to a single point then the density is a point mass and we write $h_{d,\mathbf{e}_i} = H(\{\mathbf{e}_i\})$.
- Using the result from the previous slide, we can write:

$$H(\{\mathbf{e}_i\}) = -z_i^2 \lim_{\substack{z_j \to 0 \\ j \notin \mathcal{S}}} \frac{\partial V}{\partial z_i}(\mathbf{z})$$

Example - 2-dimensional simplex

The densities on the **vertices** of the simplex are given by

$$H(\{\mathbf{e}_1\}) = -z_1^2 \lim_{z_2 \to 0} \frac{\partial V}{\partial z_1}(z_1, z_2) \quad \text{and} \quad H(\{\mathbf{e}_2\}) = -z_2^2 \lim_{z_1 \to 0} \frac{\partial V}{\partial z_2}(z_1, z_2).$$

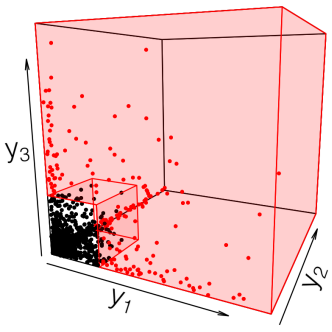
The density in the **interior** of the simplex is

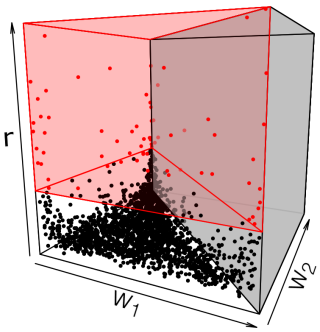
$$h_{2,\{1,2\}}(\omega) = -\frac{\partial^2 V}{\partial z_1 \partial z_2}(\omega, 1-\omega).$$

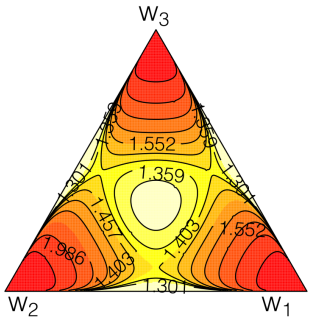


Write down the expression for the densities on each subspace of the simplex when $d = 3$.

Angular densities

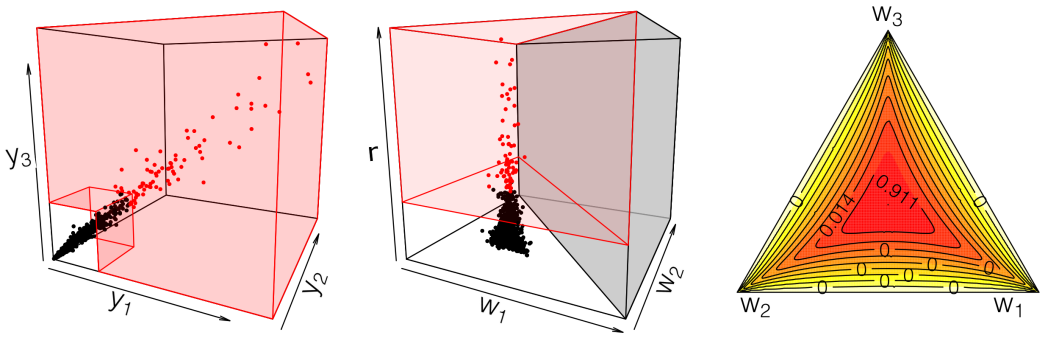






Weak dependence: angular density puts mass towards the vertices.

Angular densities



Strong dependence: angular density concentrated towards the centre of the simplex.

4. Quantifying and Representing Extremal Dependence

Recap

So far we have established that **the extremal dependence is expressed through the exponent function** V, which is a map from $\mathbb{R}_+^d$ to $(0, \infty)$ with the following properties:

- [Homogeneity of order -1] $V(a\mathbf{z}) = aV(\mathbf{z}), \forall a > 0$.
- [Convexity] $V(a\mathbf{z} + (1-a)\mathbf{z}') \leq aV(\mathbf{z}) + (1-a)V(\mathbf{z}'), \forall a \in [0,1], \mathbf{z}, \mathbf{z}' \in \mathbb{R}_+^d$.
- [Ranges between complete dependence and complete independence] $\max\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right) \leq V(\mathbf{z}) \leq \frac{1}{z_1} + \ldots + \frac{1}{z_d}$.

Recap

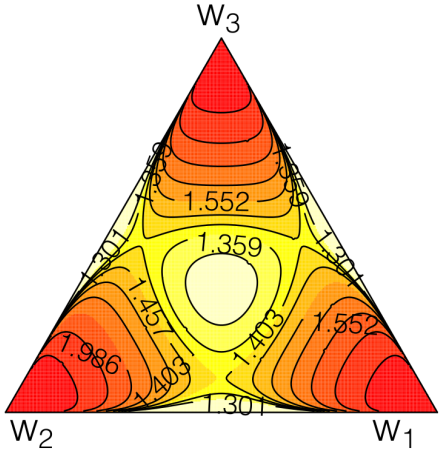
Let $Z_1, \ldots, Z_d \overset{\text{i.i.d.}}{\sim} G$. When H places mass of:

- 1 **on the centre of the simplex** $(1/d, \ldots, 1/d)$ then $Z_1 = \cdots = Z_d$ almost surely and $G(\mathbf{z}) = \exp\{\max(1/z_1, \ldots, 1/z_d)\}$.
- $1/d$ **on the vertices of the simplex** $\mathbf{e}_j, \forall j \in \{1, \ldots, d\}$ then $Z_1, \ldots, Z_d$ are independent and $G(\mathbf{z}) = \exp\{1/z_1 + \cdots + 1/z_d\}$.

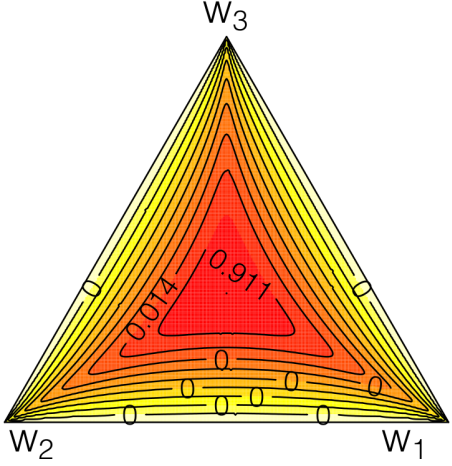
The angular density

One easy way to visualise the extremal dependence is to plot the angular density itself!

Weak dependence



Strong dependence



Quantifying the extremal dependence

Clearly graphical summaries of extremal dependence are ok up until $d = 3$. We (re)-introduce some **numerical summaries** to get a better understanding in higher dimensions.

Definition[Smith, 1990]

The *extremal coefficient* is given by

$$\vartheta = V(1, \ldots, 1) = dA\left(\frac{1}{d}, \ldots, \frac{1}{d}\right) = d \int_{\mathbb{W}} \max_{j=1,\ldots d}(\omega_j) dH(\boldsymbol{\omega}).$$

- ϑ represents the number of independent components of the random vector $\mathbf{Y}$.
- $\vartheta \in [1, d]$ with bounds indicating complete dependence/independence.



Can you prove the last comment using a result previously stated?

Example - Independence when $d = 2$

We have seen that in that case H puts mass $1/2$ at $\omega = 0$ and $\omega = 1$. Therefore we have

$$\begin{aligned}\vartheta &= 2\int_0^1 \max(\omega, 1-\omega) dH(\boldsymbol{\omega}) \\ &= 2\left(\max(0,1) \times \frac{1}{2} + \max(1,0) \times \frac{1}{2}\right) \\ &= 2.\end{aligned}$$

Alternatively, we can also recall that in case of complete independence $V(z_1, z_2) = 1/z_1 + 1/z_2$. Therefore $\vartheta = V(1, \ldots, 1) = 2$.

Example - Complete dependence when $d = 2$

We have seen that in that case H puts mass 1 at $\omega = 1/2$. Therefore we have

$$\begin{aligned}\vartheta &= 2\int_0^1 \max(\omega, 1-\omega) dH(\omega) \\ &= 2\left(\max\left(\frac{1}{2}, \frac{1}{2}\right) \times 1\right) \\ &= 1.\end{aligned}$$

Alternatively, we can also recall that in case of complete dependence $V(z_1, z_2) = \max(1/z_1, 1/z_2)$. Therefore $\vartheta = V(1, \ldots, 1) = 1$.



Reproduce the above examples when $d = 3$.

Definition [Nikoloulopoulos et al., 2009]

For any $\mathbf{z} > \mathbf{0}$ and failure region

$$\mathcal{F}_{\mathbf{z}} = \left(\mathbf{v} \in \mathbb{R}_+^d \backslash \{\mathbf{0}\} : v_1 > z_1 \text{ and } \ldots \text{ and } v_d > z_d\right),$$

the **tail dependence function** is defined by

$$R(\mathbf{z}) := \eta\left\{\left(\mathbf{v} \in \mathbb{R}_+^d \backslash \{\mathbf{0}\} : v_1 > z_1 \text{ and } \ldots \text{ and } v_d > z_d\right)\right\} \equiv \eta\left\{\mathcal{F}_{\mathbf{z}}\right\}.$$

This counts the number of observations that fall in the failure region (all the coordinates are simultaneously large).

Quantifying the extremal dependence



The tail dependence function is related to the exponent function by the inclusion-exclusion principle through

$$R(\mathbf{z}) = d \int_{\mathbb{W}} \min_{j=1,\ldots,d} \left(\frac{w_j}{z_j} \right) \mathrm{d}H(\boldsymbol{\omega}).$$

Prove this result in the case $d = 2$ and 3.

Quantifying the extremal dependence

Another summary of the dependence between the elements of $\mathbf{Y}$ is defined as follows.

Definition [Joe, 1997, Chap. 2; Li, 2009]

The **coefficient of upper tail dependence** is given by

$$\chi = R(1, \ldots, 1) = d \int_{\mathbb{W}} \min_{j=1,\ldots,d} (w_j) \, \mathrm{d}H(\boldsymbol{\omega}).$$

It measures the strength of dependence in the tail distribution of $\mathbf{Y}$ or in other terms the probability that all the components of $\mathbf{Y}$ are simultaneously large.

Quantifying the extremal dependence

Example - Independence/dependence when $d = 2$

In the case of **independence**, we know that H puts mass $1/2$ at $\omega = 0$ and $\omega = 1$. Therefore we have

$$\chi = 2\int_0^1 \min(\omega, 1-\omega) dH(\boldsymbol{\omega}) = 2\left(\min(0,1) \times \frac{1}{2} + \min(1,0) \times \frac{1}{2}\right) = 0.$$

In the case of **complete dependence**, we know that H puts mass 1 at $\omega = 1/2$. Therefore we have

$$\vartheta = 2\int_0^1 \min(\omega, 1-\omega) dH(\boldsymbol{\omega}) = 2\left(\min\left(\frac{1}{2}, \frac{1}{2}\right) \times 1\right) = 1.$$



Reproduce the above example when $d = 3$.

5. Parametric Models for Extremal Dependence



Parametric models for extremal dependence

So far we have seen that:

- Exponent and tail dependence functions depend on the angular measure.
- The angular measure is **not unique**.
- **Any measure** that satisfies the first moment condition is suitable.

In principle it is insufficient to use a parametric model for the d.f. of the angular measure.

However we will see that some classes of parametric models can be useful for applications. We define some of the most popular models.

Asymmetric logistic

Recall that the exponent function is given by

$$V(\mathbf{z};\boldsymbol{\theta}) = \sum_{\mathcal{S}\in\mathbb{S}} \left\{ \sum_{j\in\mathcal{S}} \left(\frac{\beta_{j,\mathcal{S}}}{z_j} \right)^{\alpha_{\mathcal{S}}} \right\}^{1/\alpha_{\mathcal{S}}}, \quad \mathbf{z} \in \mathbb{R}_+^d,$$

where $\boldsymbol{\theta} = \{\alpha_{\mathcal{S}}, \boldsymbol{\beta}_{\mathbf{j},\mathcal{S}}\}_{\mathcal{S}\in\mathbb{S}}$, $\alpha_{\mathcal{S}} \geq 1$, $\beta_{\mathcal{S}} = \tau_{\mathcal{S}} / \sum_{\mathcal{S}\in\mathbb{S}_j} \tau_{\mathcal{S}}$ and $\beta_{j,\mathcal{S}} = 0$ if $j \notin \mathcal{S}$ and for $j \in \{1,\ldots,d\}$, $0 \leq \beta_{j,\mathcal{S}} \leq 1$ and $\sum_{\mathcal{S}\in\mathbb{S}} \beta_{j,\mathcal{S}} = 1$.

$\beta_{j,\mathcal{S}}$ represents the probability that the maximum value value observed at site j is attributed to a storm event involving the sites of $\mathcal{S}$.

The number of model parameters is $2^{d-1}(d+2) - (2d+1)$.

Angular density

The general expression is given by

$$h_{d,\mathcal{S}}(\boldsymbol{\omega};\boldsymbol{\theta}) = \prod_{i=1}^{k-1}(i\alpha_{\mathcal{S}}-1)\prod_{j\in\mathcal{S}}\beta_{j,\mathcal{S}}^{\alpha_{\mathcal{S}}}\omega_j^{-(\alpha_{\mathcal{S}}+1)}\left\{\sum_{j\in\mathcal{S}}\left(\frac{\beta_{j,\mathcal{S}}}{\omega_j}\right)^{\alpha_{\mathcal{S}}}\right\}^{1/\alpha_{\mathcal{S}}-k}, \quad \boldsymbol{\omega}\in\mathbb{W}_{d,\mathcal{S}}.$$

- For $\mathcal{S}=\{1,\ldots,d\}$, $\alpha_{\mathcal{S}}=\alpha$, $\beta_{j,\mathcal{S}}=\beta_j$, the angular density in the **interior** of the simplex is

$$h_d(\boldsymbol{\omega};\boldsymbol{\theta}) = \prod_{i=1}^{d-1}(i\alpha-1)\prod_{j\in\mathcal{S}}\beta_j^{\alpha}\omega_j^{-(\alpha+1)}\left\{\sum_{j\in\mathcal{S}}\left(\frac{\beta_j}{\omega_j}\right)^{\alpha}\right\}^{1/\alpha-d}, \quad \boldsymbol{\omega}\in\mathbb{W}$$

- For $\mathcal{S}=\{j\}, j=1,\ldots,d$, the point mass at the **vertex** of the simplex is $h_{d,\{j\}}=\beta_j$,

Asymmetric logistic

Example - $d = 2$

The model has $2(2+2) - (2 \times 2 + 1) = 3$ parameters.

Here $\mathbb{S} = \{\{1\}, \{2\}, \{1,2\}\}$ and

- the **vertices** are $\mathbb{W}_{2,\{1\}} = \{(1,0)\}$ and $\mathbb{W}_{2,\{2\}} = \{(0,1)\}$.
- the **interior** is $\mathbb{W}_{2,\{1,2\}} = \{(\omega, 1-\omega), \omega \in (0,1)\}$.

The conditions on the parameters are

- $\beta_{1,\{1\}} + \beta_{1,\{2\}} + \beta_{1,\{1,2\}} = \beta_{1,\{1\}} + \beta_{1,\{1,2\}} = 1 \Longrightarrow \beta_{\mathbf{1},\{\mathbf{1}\}} = \mathbf{1} - \beta_{\mathbf{1}}$
- $\beta_{2,\{1\}} + \beta_{2,\{2\}} + \beta_{2,\{1,2\}} = \beta_{2,\{2\}} + \beta_{2,\{1,2\}} = 1 \Longrightarrow \beta_{\mathbf{2},\{\mathbf{2}\}} = \mathbf{1} - \beta_{\mathbf{2}}$

Vector of parameters is $\theta = (\alpha, \beta_1, \beta_2)$.

Asymmetric logistic

Example - $d = 2$

The angular density in the **interior** of the simplex is

$$h'_{2,\{1,2\}}(\omega) = (\alpha-1)(\beta_1\beta_2)^\alpha \left(\omega(1-\omega)\right)^{-(\alpha+1)} \left\{ \left(\frac{\beta_1}{\omega}\right)^\alpha + \left(\frac{\beta_2}{1-\omega}\right)^\alpha \right\}^{1/\alpha-2}$$
$$= (\alpha-1)(\beta_1\beta_2)^\alpha \left(\omega(1-\omega)\right)^{\alpha-2} \left\{ (\beta_1(1-\omega))^\alpha + (\beta_2\omega)^\alpha \right\}^{1/\alpha-2}.$$

The point masses at the **vertices** of the simplex are

$$h'_{2,\{1\}} = 1-\beta_1 \quad \text{and} \quad h'_{2,\{2\}} = 1-\beta_2$$

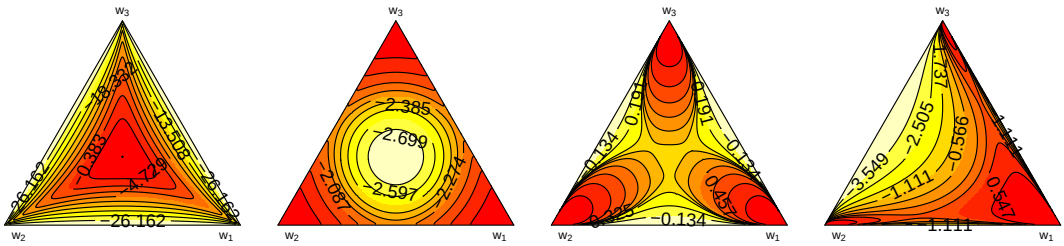


Plot the angular density of the asymmetric logistic model for various values of θ.

Asymmetric logistic

Example - $d = 3$

The vector of parameters is $\theta = (\alpha, \beta_1, \beta_2, \beta_3)$.



Left to right: $\theta = (5.75, 0.5, 0.5, 0.5), (1.01, 0.9, 0.9, 0.9), (1.25, 0.5, 0.5, 0.5)$ and $(1.4, 0.7, 0.15, 0.15)$.



What impact do each parameter have on the dependence structure?

Asymmetric logistic



Consider the $d = 3$ setting and the the combinations of parameter values considered in the previous example.

1. What are the masses on the vertices and edges of the simplex?
2. Calculate the value of the exponent and upper tail dependence coefficients.
3. What are some noticeable features of the model when $\beta_1 = \beta_2 = \beta_3 = 1$?

Hüsler-Reiss (Hüsler & Reiss, 1989)

Recall that the exponent function is given by

$$V(\mathbf{z};\boldsymbol{\theta}) = \sum_{j=1}^{d} \frac{1}{z_j} \Phi_{d-1}\left\{ \left(\lambda_{i,j} + \frac{\log z_i/z_j}{2\lambda_{i,j}} \right)_{i\in\{1,\ldots,d\}\setminus\{j\}} ; \bar{\Lambda}_j \right\},$$

where Φ_{d-1} is the cdf of $d-1$ dimensional Gaussian distribution with partial correlation $\bar{\Lambda}_j$ with elements

$$\lambda_{k,i;j} = \frac{\lambda_{k,j}^2 + \lambda_{i,j}^2 - \lambda_{k,i}^2}{2\lambda_{k,j}\lambda_{i,j}}, \quad k,i \in \{1,\ldots,d\}\setminus\{j\}; k < i.$$

These are define in row order the upper triangle (or in column order the lower triangle) of the matrix and symmetry needs to be used to complete it.

The parameter vector is $\boldsymbol{\theta} = \left(\{\lambda_{i,j}\}_{i,j\in\{1,\ldots,d\}} \right)$.

$\lambda_{i,j}$: controls the dependence between the i-th and j-th elements of a vector f d extremes. $\lambda_{i,j} = 0 \Longrightarrow$ Complete dependence. $\lambda_{i,j} \to \infty \Longrightarrow$ Independence.

Angular density

The angular density in the **interior** of the simplex is

$$h'(\boldsymbol{\omega};\boldsymbol{\theta}) = \phi_{d-1}\left\{\left(\lambda_{i,1} + \frac{\log \omega_i/\omega_1}{2\lambda_{i,1}}\right)_{i\in\{2,\dots,d\}}; \bar{\Lambda}_1\right\}\left\{\omega_1^2 \prod_{i=2}^{d}(\omega_i 2\lambda_{i,1})\right\}^{-1}, \quad \boldsymbol{\omega} \in \mathbb{W}.$$

There is **no mass** on any other **sub-space of the simplex**.

Example - $d = 2$

There is only 1 parameter given by $\theta = \lambda_{1,2} = \lambda > 0$.

The angular density in the **interior** of the simplex is

$$h'_{2,\{1,2\}}(\omega) = \phi\left\{\lambda + \frac{\log((1-\omega)/\omega)}{2\lambda}\right\}\left\{\omega^2(1-\omega)2\lambda\right\}^{-1}, \quad 0 < \omega < 1.$$

The point masses at the **vertices** of the simplex are, by construction,

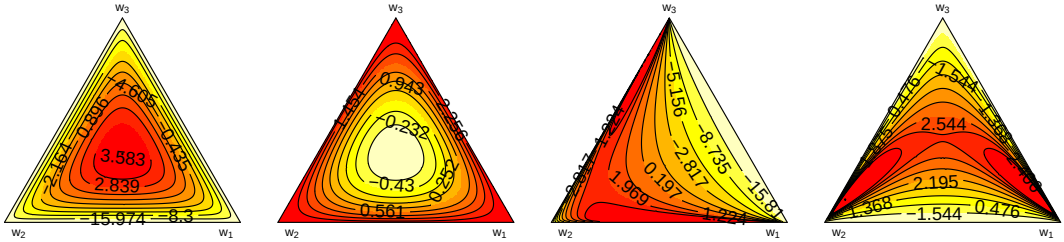
$$h'_{2,\{1\}} = 0 \quad \text{and} \quad h_{2,\{2\}} = 0.$$



Plot the angular density of the Hüsler-Reiss model for various values of θ.

Example - $d = 3$

The vector of parameters is $\theta = (\lambda_{1,2}, \lambda_{1,3}, \lambda_{2,3})$.



Left to right: $\theta = (0.3, 0.3, 0.3), (1.4, 1.4, 1.4), (1.7, 0.7, 1.1)$ and $(0.52, 0.71, 0.52)$.



What impact do each parameter have on the dependence structure?

Hüsler-Reiss



Consider the $d = 3$ setting and the the combinations of parameter values considered in the previous example.

Calculate the value of the exponent and upper tail dependence coefficients.

Extremal-t (Nikoloulopoulos et al., 2009)

Recall that the exponent function is given by

$$V(\mathbf{z};\boldsymbol{\theta}) = \sum_{j=1}^{d} \frac{1}{z_j} T_{d-1;\nu+1}\left\{ \left(\sqrt{\frac{\nu+1}{1-\rho_{i,j}^2}} \left(\left(\frac{z_i}{z_j} \right)^{1/\nu} - \rho_{i,j} \right) \right)_{i \in \{1,\ldots,d\} \setminus \{j\}} ; \bar{\Sigma}_j \right\},$$

where $T_{d-1;\nu+1}$ is the cdf of $d-1$ dimensional Student-t with $\nu+1$ DoF and partial correlation $\bar{\Sigma}_j$ with elements

$$\sigma_{k,i;j} = \frac{\rho_{k,i} - \rho_{k,j}\rho_{i,j}}{\sqrt{(1-\rho_{k,j}^2)(1-\rho_{i,j}^2)}}.$$

The parameter vector is $\boldsymbol{\theta} = \left(\{\rho_{\mathbf{i},\mathbf{j}}\}_{\mathbf{i},\mathbf{j} \in \{\mathbf{1},\ldots,\mathbf{d}\}}, \nu \right)$.

$\rho_{i,j}$: **controls the dependence between the i-th and j-th elements.**

ν: **controls the overall dependence among all the variables**.

Angular density

The angular density in the **interior** of the simplex is

$$h'(\boldsymbol{\omega};\boldsymbol{\theta}) = \frac{t_{d-1;\nu+1}\left\{\left(\sqrt{\frac{\nu+1}{1-\rho_{i,1}^2}}\left(\left(\frac{\omega_i}{\omega_1}\right)^{1/\nu}-\rho_{i,1}\right)\right)_{i\in\{2,\dots,d\}};\bar{\Sigma}_1\right\}}{\nu^{d-1}\omega_1^{d+1}\left\{\prod_{i=2}^{d}\sqrt{\frac{\nu+1}{1-\rho_{i,1}^2}}\left(\frac{\omega_i}{\omega_1}\right)^{1-1/\nu}\right\}^{-1}},$$

where $t_{d-1;\nu+1}$ is the pdf of the $d-1$ dimensional Student-t distribution with $\nu+1$ DoF and partial correlation $\bar{\Sigma}_1$.

The angular density in the **vertices** of the simplex is

$$h'_{d,\{j\}}(\boldsymbol{\theta}) = T_{d-1,\nu+1}\left\{\left(-\rho_{i,j}\sqrt{\frac{\nu+1}{1-\rho_{i,j}^2}}\right)_{i\in\{1,\dots,d\}\setminus\{j\}};\bar{\Sigma}_j\right\}.$$

Extremal-t

Example - $d = 2$

There are 2 parameters: $\boldsymbol{\theta} = (\rho_{1,2}, \nu) = (\rho, \nu)$ where $\rho \in [-1, 1]$ and $\nu > 0$.

The angular density in the **interior** of the simplex is

$$h'_{2,\{1,2\}}(\omega) = \frac{t_{\nu+1}\left\{\left(\sqrt{\frac{\nu+1}{1-\rho^2}}\left(\left(\frac{1-\omega}{\omega}\right)^{1/\nu} - \rho\right)\right)\right\}}{\nu\omega^3\left\{\sqrt{\frac{\nu+1}{1-\rho^2}}\left(\frac{1-\omega}{\omega}\right)^{1-1/\nu}\right\}^{-1}}, \quad 0 < \omega < 1.$$

The point masses at the **vertices** of the simplex are, by construction,

$$h'_{2,\{1\}} = T_{\nu+1}\left\{-\rho\sqrt{\frac{\nu+1}{1-\rho^2}}\right\} \quad \text{and} \quad h'_{2,\{2\}} = T_{\nu+1}\left\{-\rho\sqrt{\frac{\nu+1}{1-\rho^2}}\right\}.$$

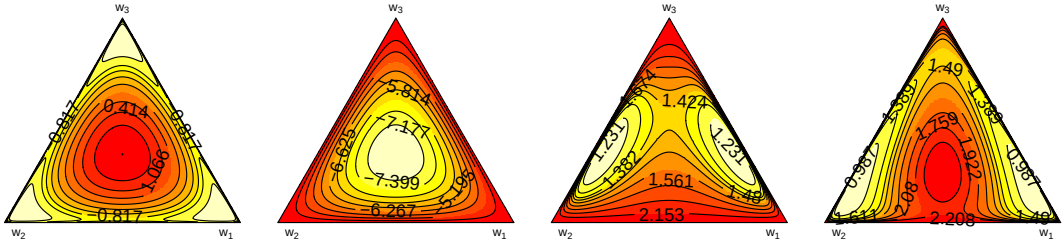


Plot the angular density of the Extremal-t model for various values of θ.

Extremal-t

Example - $d = 2$

The vector of parameters is $\theta = (\rho_{1,2}, \rho_{1,3}, \rho_{2,3}, \nu)$.



Left to right: $\theta = (0.95, 0.95, 0.95, 2), (-0.3, -0.3, -0.3, 5), (0.52, 0.71, 0.52, 3)$ and $(0.52, 0.71, 0.52, 2)$.



What impact do each parameter have on the dependence structure?

Extremal-t



Consider the $d = 3$ setting and the the combinations of parameter values considered in the previous example.

Calculate the value of the exponent and upper tail dependence coefficients.

Tilted Dirichlet (Coles & Tawn, 1991)

Idea: have less parameters & define a model with angular density concentrating in the interior of the simplex.

Construction

1. Take $h' : \mathbb{W} \to [0, \infty)$ s.t. $m_j = \int_{\mathbb{W}} v_j h'(\mathbf{v}) \mathrm{d}\mathbf{v} < \infty$, $\forall j = 1, \ldots, d$.
2. A valid angular density is obtained as the function

$$h(\boldsymbol{\omega}) = d^{-1} \left(m_1\omega_1 + \cdots + m_d\omega_d\right)^{-(d+1)} h' \left(\frac{\mathbf{m}\boldsymbol{\omega}}{m_1\omega_1 + \cdots + m_d\omega_d} \right),$$

with first moments $1/d$, mass centred at $(1/d, \ldots, 1/d)$ and integrating to 1.

3. If h' is the Dirichlet density, then for $\boldsymbol{\theta} = \{\alpha_j > 0\}$, $\boldsymbol{\omega} \in \mathbb{W}$,

$$h(\boldsymbol{\omega}; \boldsymbol{\theta}) = \frac{\Gamma\left(\sum_{j\in\{1,\ldots,d\}} \alpha_j + 1\right)}{d\left(\sum_{j\in\{1,\ldots,d\}} \alpha_j\omega_j\right)^{d+1}} \prod_{j=1}^{d} \frac{\alpha_j}{\Gamma(\alpha_j)} \left(\frac{\alpha_j\omega_j}{\sum_{j\in\{1,\ldots,d\}} \alpha_j\omega_j} \right)^{\alpha_j - 1}.$$

Tilted Dirichlet

Comments

- The angular density is asymmetric, symmetry occurs when $\alpha_1 = \cdots = \alpha_d$.
- Extremes are **independent** when $\alpha_j \to 0, \forall j = 1, \ldots, d$.
- Extremes are **completely dependent** when $\alpha_j \to \infty, \forall j = 1, \ldots, d$.
- The exponent function cannot be computed analytically but we can use numerical integration.



The parameters are difficult to interpret!

Coles & Tawn (1994) introduce $r_1 = (\alpha_i - \alpha_j)/2$ and $r_2 = (\alpha_i + \alpha_j)/2$ to measure the asymmetry and intensity of the dependence between pairs.

Tilted Dirichlet

Example - $d = 2$

Vector of parameters is $\boldsymbol{\theta} = (\alpha_1, \alpha_2) > (0, 0)$.

The angular density in the **interior** of the simplex is

$$h_{2,\{1,2\}}(\omega) = \frac{\Gamma(\alpha_1 + \alpha_2 + 1)}{2(\alpha_1\omega + \alpha_2(1-\omega))^{\alpha_1+\alpha_2+1}} \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \omega^{\alpha_1-1}(1-\omega)^{\alpha_2-1}$$

The point masses at the **vertices** of the simplex are, by construction,

$$h_{2,\{1\}} = 0 \quad \text{and} \quad h_{2,\{2\}} = 0.$$

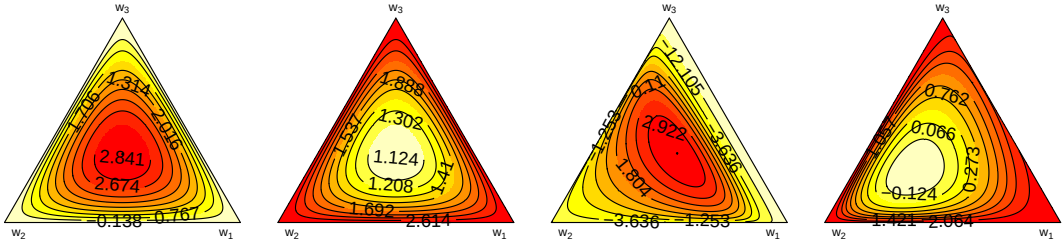


Plot the angular density of the tilted Dirichlet model for various values of θ.

Tilted Dirichlet

Example - $d = 2$

The vector of parameters is $\theta = (\alpha_1, \alpha_2, \alpha_3)$.



Left to right: $\theta = (2, 2, 2), (0.5, 0.5, 0.5), (2, 2.5, 30)$ and $(0.1, 0.25, 0.95)$.



What impact do each parameter have on the dependence structure?

Pairwise Beta (Cooley et al., 2010)

Idea: More easily interpretable parameters using a geometric approach.

Construction

1. Consider the symmetric pairwise beta function between any pairs of $\boldsymbol{\omega}$

$$h^*(\omega_i, \omega_j) = \frac{\Gamma(2\beta_{i,j})}{\Gamma(\beta_{i,j})^2} \left(\frac{\omega_i}{\omega_i + \omega_j}\right)^{\beta_{i,j}-1} \left(\frac{\omega_j}{\omega_i + \omega_j}\right)^{\beta_{i,j}-1}, \quad i, j \in \{1, \ldots, d\},$$

with first moments $1/d$, mass centred at $(1/d, \ldots, 1/d)$ and integrating to 1.

2. Summing together all the $d(d-1)/2$ possible pairs of variables gives

$$h(\boldsymbol{\omega}; \boldsymbol{\theta}) = \frac{2(d-3)!\Gamma(\alpha d + 1)}{d(d-1)\Gamma(2\alpha + 1)\Gamma(\alpha(d-2))} \sum_{\substack{i,j \in \{1,\ldots,d\} \\ i<j}} h(\omega_i, \omega_j), \quad \boldsymbol{\omega} \in \mathbb{W},$$

where $h(\omega_i, \omega_j) = (\omega_i + \omega_j)^{2\alpha-1} [1 - (\omega_i + \omega_j)]^{\alpha(d-2)-d+2} h^*(\omega_i, \omega_j)$.

Pairwise Beta

Comments

- There are $d(d-1)/2+1$ parameters given in vector form by

$$\boldsymbol{\theta} = \left(\alpha, \{\beta_{\mathbf{i,j}}\}_{\mathbf{i,j}\in\{\mathbf{1},\ldots,\mathbf{d}\},\mathbf{i}<\mathbf{j}}\right) > (0,\ldots,0).$$

$\beta_{i,j}$: **controls the level of dependence between the i-th and j-th components**. If $\beta_{i,j}$ increases, the dependence increases.

α: **controls the overall dependence between all variables**. If α increases, the dependence increases.

- The function h^* is introduced to guarantee that the weak and strong dependence regimes can be attained.
- The exponent function needs to be evaluated numerically.

Tilted Dirichlet

Example - $d = 2$

There are $2(2-1)/2+1 = 2$ parameters given in the vector $\boldsymbol{\theta} = (\alpha, \beta_{1,2}) > (0,0)$.
The angular density in the **interior** of the simplex is

$$\begin{aligned} h_{2,\{1,2\}}(\omega) &= h^*(\omega, 1-\omega) \\ &= \frac{\Gamma(2\beta_{1,2})}{\Gamma(\beta_{1,2})^2} \omega^{\beta_{1,2}-1} (1-\omega)^{\beta_{1,2}-1}. \end{aligned}$$

The point masses at the **vertices** of the simplex are, by construction,

$$h_{2,\{1\}} = 0 \quad \text{and} \quad h_{2,\{2\}} = 0.$$

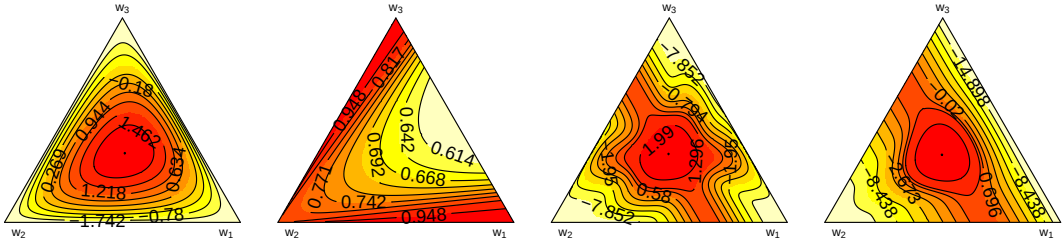


Plot the angular density of the Pairwise Beta model for various values of θ.

Tilted Dirichlet

Example - $d = 2$

The vector of parameters is $\theta = (\alpha, \beta_{1,2}, \beta_{1,3}, \beta_{2,3})$.



{Left to right: $\theta = (4, 2, 2, 2), (0.5, 1, 1, 1), (1, 2, 4, 15)$ and $(1, 10, 10, 10)$.}



What impact do each parameter have on the dependence structure?

References



Sources and recommending reading

1. **[BGST04]** Beirlant, J., Goegebeur, Y., Segers, J. & Teugels J. (2004) *Statistics of Extremes: Theory and Applications*. John Wiley & Sons. Chapter 8 & 9

2. **[C01]** Coles, S. (2001) An Introduction to Statistical Modeling of Extreme Values. Springer. Chapter 7.

3. **[BP16]** Béranger, B. & S. A. Padoan (2016) Extremal Dependence Models. In *Extreme Value Modelling and Risk Analysis: Methods and Applications.* Eds. Dey and J. Yan. Chapman & Hall/CRC Press.