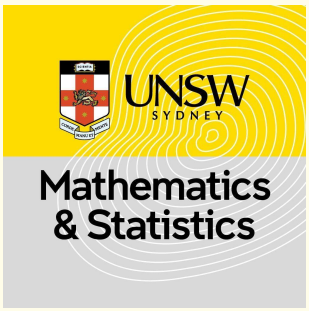


MATH5805
A Modern Introduction to Extreme Value Theory
Term 1, 2025

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Week 8 - Tutorial Solutions

The exercises given here are directly following those from Week 7.

Exercise 1

Express the tail dependence function R with respect to the exponent function V for dimension $d = 3$.

Solution:
Let $\boldsymbol{z} \in \mathbb{R}^3_+$. Let $A = \{\boldsymbol{x} \in \mathbb{R}^3_+ : x_1 > z_1\}$, $B = \{\boldsymbol{x} \in \mathbb{R}^3_+ : x_2 > z_2\}$, and $C = \{\boldsymbol{x} \in \mathbb{R}^3_+ : x_3 > z_3\}$. We denote the *failure region* by

$$\mathcal{F}_{\boldsymbol{z}} = A \cap B \cap C.$$

Let $\mathcal{A}_{\boldsymbol{z}} = A \cup B \cup C$ Then, by the inclusion-exclusion principle, we have that

$$\eta\mathcal{A}_{\boldsymbol{z}} = \eta A + \eta B + \eta C - \eta\{A \cap B\} - \eta\{A \cap C\} - \eta\{B \cap C\} + \eta\mathcal{F}_{\boldsymbol{z}},$$

where $\eta\mathcal{A}_{\boldsymbol{z}} =: V(\boldsymbol{z})$ and $\eta\mathcal{F}_{\boldsymbol{z}} =: R(\boldsymbol{z})$. The measure on each margin is the reciprocal of the point, and hence

$$\eta A = \frac{1}{z_1}, \eta B = \frac{1}{z_2}, \text{ and } \eta C = \frac{1}{z_3}.$$

Now, observe that

$$\eta A \cup B = \int_{\mathbb{W}_2} \frac{\omega}{x_1} \vee \frac{1-\omega}{x_2} \, \mathrm{d}H(\omega) = \int_{\mathbb{W}_3} \max\left\{\frac{\omega_1}{x_1}, \frac{\omega_2}{x_2}, \frac{1-\omega_1-\omega_2}{\infty}\right\} \mathrm{d}H(\boldsymbol{\omega}) = V(x_1, x_2, \infty),$$

where we use the convention that $1/\infty = 0$. Now

$$\eta A \cap B = \eta A + \eta B - \eta A \cup B.$$

Then

$$\eta\mathcal{A}_{\boldsymbol{z}} = V(\boldsymbol{z}) = -\frac{1}{z_1} - \frac{1}{z_2} - \frac{1}{z_3} + V(z_1, z_2, \infty) + V(z_1, \infty, z_3) + V(\infty, z_2, z_3) + R(\boldsymbol{z}).$$

Rearranging for R then yields that

$$R(\boldsymbol{z}) = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} - V(z_1, z_2, \infty) - V(z_1, \infty, z_3) - V(\infty, z_2, z_3) + V(\boldsymbol{z}).$$

Exercise 2

Compute the coefficient of the upper tail dependence for dimension $d = 3$ when the random vector $\boldsymbol{Y}$ has independent and completely dependent margins.

Solution:
Let us first consider the case of perfect dependence, in which the spectral measure H submits unit mass to the point $d^{-1}(1, \ldots, 1)^\top$. We then have that

$$V(1,1,\infty) = 3\int_{\mathbb{W}} \max\left\{\frac{\omega_1}{1}, \frac{\omega_2}{1}, \frac{1-\omega_1-\omega_2}{\infty}\right\} H(\mathrm{d}\boldsymbol{\omega}) = 3 \times \frac{1}{3}\int_{\mathbb{W}} H(\mathrm{d}\boldsymbol{\omega}) = 1,$$

and identically for $V(1,\infty,1)$ and $V(\infty,1,1)$. Hence

$$\chi = R(1,1,1) = V(1,1,1) = \vartheta = 1.$$

Now consider the case of complete independence, in which the spectral measure submits mass d^{-1} to the vertices of the simplex $\mathbb{W}$. Then

$$\begin{aligned} V(1,1,\infty) &= 3\int_{\mathbb{W}} \frac{\omega_1}{1} \vee \frac{\omega_2}{1} \vee \frac{1-\omega_1-\omega_2}{\infty} H(\mathrm{d}\boldsymbol{\omega}) \\ &= 3\left(1 \times H(\boldsymbol{e}_1) + 1 \times H(\boldsymbol{e}_2) + 0 \times H(\boldsymbol{e}_3)\right) \\ &= 2, \end{aligned}$$

and identically for $V(1,\infty,1)$ and $V(\infty,1,1)$. Hence

$$\chi = R(1,1,1) = 3 - 6 + \vartheta = \vartheta - 3 = 0.$$

Exercise 3

Let $\boldsymbol{Z}$ be a random vector with unit Pareto margins and distribution function F. Show that $1 - F(n\boldsymbol{z}) \approx V(n\boldsymbol{z})$ and $\mathbb{P}(Z_1 > nz_1, \ldots Z_d > nz_d) \approx R(n\boldsymbol{z})$.

Solution:
Firstly, recall that the Pareto distribution is in the Fréchet domain of attraction. Hence, we can apply that $n(1 - F(n\boldsymbol{y})) \to -\log G(\boldsymbol{y}) = V(\boldsymbol{y})$. By homogeneity of the exponent function, we then have that $1 - F(n\boldsymbol{y}) \approx V(n\boldsymbol{y})$. Hence, for sufficiently large n and threshold, the extremal dependence of the Pareto distribution is approximated by the Fréchet distribution. We can hence surmise that the joint failure region can also be approximated by the Fréchet distribution.