UNSW SYDNEY

MATH5805 – A Modern Introduction to Extreme Value Theory

Week 8 - Multivariate Extremes: the componentwise maxima (II)

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Lecture outline

1. Introduction
2. Limit laws
3. Quantifying and representing extremal dependence
4. Parametric models for extremal dependence
5. Extremal dependence estimation
6. Application to pollution data
7. Application to wind speed data

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5. Extremal dependence estimation



Extremal dependence estimation

Since we have introduced some parametric models for the extremal dependence in the previous section, we will focus on **parametric approaches.** Note that **non-parametric approaches** exist but are out of the scope of this course.

In particular we will focus on the **Composite likelihood** techniques.

Bayesian estimation techniques exists but are also out of the scope of this course.

Likelihood

So far we have established that the **distribution function** of MEVDs is of the form

$$G(\mathbf{z};\boldsymbol{\theta}) = \exp\left\{-V(\mathbf{z};\boldsymbol{\theta})\right\}, \quad \mathbf{z} \in \mathbb{R}_+^d.$$

Therefore, the **probability density function** is obtained as

$$\begin{aligned} g(\mathbf{z};\boldsymbol{\theta}) &= \frac{\partial^d}{\partial z_1 \cdots \partial z_d} \exp\left\{-V(\mathbf{z};\boldsymbol{\theta})\right\} \\ &= \exp\left\{-V(\mathbf{z};\boldsymbol{\theta})\right\} \times \sum_{\Pi \in \mathcal{P}_d} \prod_{k=1}^{|\Pi|} -V_{\pi_k}(\mathbf{z};\boldsymbol{\theta}), \end{aligned}$$

where $\mathcal{P}_d$ is the set of all possible partitions of $\{1,\ldots,d\}$, each partition $\Pi \in \mathcal{P}_d$ has elements π_k, for $k = 1,\ldots,|\Pi|$, and $V_{\pi_k}(\cdot)$ represents the partial derivatives of $V(\cdot)$ w.r.t. π_k.

Likelihood

Example - $d = 2$

$\mathcal{P}_2 = \{(\{1\}, \{2\}), (\{1, 2\})\}$

- When $\Pi = (\{1\}, \{2\})$, $k = 1, 2$ and $\pi_1 = \{1\}, \pi_2 = \{2\}$.
- When $\Pi = \{1, 2\}$, $k = 1$ and $\pi_1 = \{1, 2\}$.

Therefor the density of a bivariate extreme value distribution is given by

$$g(z_1, z_2; \boldsymbol{\theta}) = \exp\{-V(z_1, z_2; \boldsymbol{\theta})\} \times (V_1 V_2 - V_{1,2})$$

Example - $d = 3$

$\mathcal{P}_3 = \{(\{1\},\{2\},\{3\}),(\{1\},\{2,3\}),(\{2\},\{1,3\}),(\{3\},\{1,2\}),(\{1,2,3\})\}$

- When $\Pi = (\{1\},\{2\},\{3\})$, $k = 1,2,3$ and $\pi_1 = \{1\}, \pi_2 = \{2\}, \pi_3 = \{3\}$.
- When $\Pi = (\{1\},\{2,3\})$, $k = 1,2$ and $\pi_1 = \{1\}, \pi_2 = \{2,3\}$.
- When $\Pi = (\{2\},\{1,3\})$, $k = 1,2$ and $\pi_1 = \{2\}, \pi_2 = \{1,3\}$.
- When $\Pi = (\{3\},\{1,2\})$, $k = 1,2$ and $\pi_1 = \{3\}, \pi_2 = \{1,2\}$.
- When $\Pi = \{1,2,3\}$, $k = 1$ and $\pi_1 = \{1,2,3\}$.

Therefor the density of a trivariate extreme value distribution is given by

$$g(z_1,z_2,z_3;\boldsymbol{\theta}) = \exp\{-V(z_1,z_2,z_3;\boldsymbol{\theta})\} \times (-V_1V_2V_3 + V_1V_{2,3} + V_2V_{1,3} + V_3V_{1,2} - V_{1,2,3})$$

Likelihood

The likelihood function to maximise w.r.t. $\boldsymbol{\theta}$ is

$$L(\mathbf{z}_1, \ldots, \mathbf{z}_n; \boldsymbol{\theta}) = \prod_{i=1}^{n} \left(\exp\{-V(\mathbf{z}_i; \boldsymbol{\theta})\} \sum_{\Pi \in \mathcal{P}_d} \prod_{k=1}^{|\Pi|} -V_{\pi_k}(\mathbf{z}_i; \boldsymbol{\theta}) \right).$$



* One can quickly see that the number of terms in the likelihood increases very fast with d (Cardinality of $\mathcal{P}_d$ is the d-th Bell number).

** The partial derivatives of $V(\cdot)$ are difficult to compute for large d.

Composite Likelihood

Rather than using the full information, an **approximation** of the joint density using **pairs**, triplets, etc.

Definition

The **pairwise composite likelihood** is written as

$$CL(\mathbf{z};\boldsymbol{\theta}) = \prod_{q \in \mathcal{Q}_d^2} L\left(\mathbf{z}^{(q)};\boldsymbol{\theta}\right),$$

where $\mathcal{Q}_d^2$ represents the set of all possible subsets of size 2 of $\{1,\ldots,d\}$ and $\mathbf{z}^{(q)}$ is a 2-dimensional subvector of $\mathbf{z} \in \mathbb{R}_+^d$.

Composite Likelihood

In other words, the pairwise composite likelihood is the product of bivariate likelihood functions at each of the possible pairs.

The pairwise composite likelihood for MEVDs takes the form

$$CL(\mathbf{z}_1, \ldots, \mathbf{z}_n; \boldsymbol{\theta}) = \prod_{i=1}^{n} \prod_{q \in \mathcal{Q}_d^2} \left(\exp\left\{ -V(\mathbf{z}_i^{(q)}; \boldsymbol{\theta}) \right\} \sum_{\Pi \in \mathcal{P}_q} \prod_{k=1}^{|\Pi|} -V_{\pi_k}(\mathbf{z}_i^{(q)}; \boldsymbol{\theta}) \right),$$

where $\mathcal{P}_q$ is the set of all possible partitions of q.

Example - $d = 2$

In that case $\mathcal{Q}_2^2 = \{\{1, 2\}\}$

Therefore q can only take value $\{1, 2\}$, implying that $\mathcal{P}_q = \{(\{1\}, \{2\}), (\{1, 2\})\}$ and

$$\begin{aligned} L\left(\mathbf{z}^{(q)}; \boldsymbol{\theta}\right) &= \exp\{-V(z_1, z_2; \boldsymbol{\theta})\} \times (V_1(z_1, z_2; \boldsymbol{\theta}) V_2(z_1, z_2; \boldsymbol{\theta}) - V_{1,2}(z_1, z_2; \boldsymbol{\theta})) \\ &= L(\mathbf{z}; \boldsymbol{\theta}) \end{aligned}$$

The pairwise composite likelihood is the regular likelihood function.

Example - $d = 3$

In that case $\mathcal{Q}_3^2 = \{\{1,2\},\{1,3\},\{2,3\}\}$

- For $q = \{1,2\}$, $\mathcal{P}_q = \{(\{1\},\{2\}),(\{1,2\})\}$ and

$$L\left(\mathbf{z}^{(q)};\boldsymbol{\theta}\right) = \exp\{-V(z_1,z_2;\boldsymbol{\theta})\} \times (V_1(z_1,z_2;\boldsymbol{\theta})V_2(z_1,z_2;\boldsymbol{\theta}) - V_{1,2}(z_1,z_2;\boldsymbol{\theta}))$$

- For $q = \{1,3\}$, $\mathcal{P}_q = \{(\{1\},\{3\}),(\{1,3\})\}$ and

$$L\left(\mathbf{z}^{(q)};\boldsymbol{\theta}\right) = \exp\{-V(z_1,z_3;\boldsymbol{\theta})\} \times (V_1(z_1,z_3;\boldsymbol{\theta})V_3(z_1,z_3;\boldsymbol{\theta}) - V_{1,3}(z_1,z_3;\boldsymbol{\theta}))$$

- For $q = \{2,3\}$, $\mathcal{P}_q = \{(\{2\},\{3\}),(\{2,3\})\}$ and

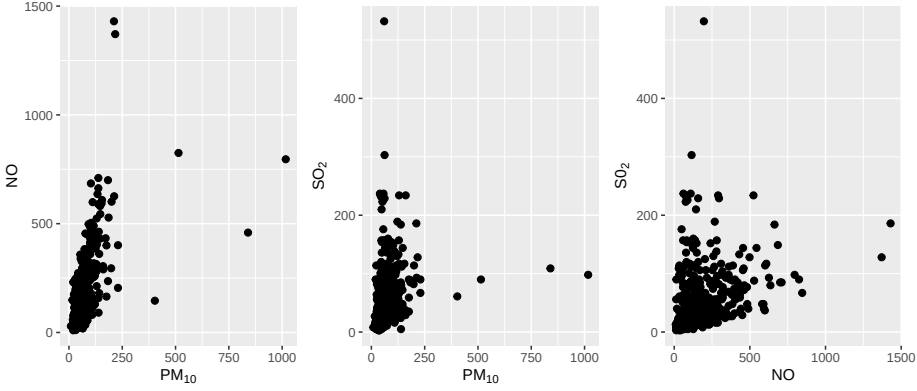
$$L\left(\mathbf{z}^{(q)};\boldsymbol{\theta}\right) = \exp\{-V(z_2,z_3;\boldsymbol{\theta})\} \times (V_2(z_2,z_3;\boldsymbol{\theta})V_3(z_2,z_3;\boldsymbol{\theta}) - V_{2,3}(z_2,z_3;\boldsymbol{\theta}))$$

6. Application to pollution data



Application to pollution data

We are going back to the example of the pollution levels in Leeds, UK with a focus on PM_{10}, NO and SO_2. Let's recall that the unit-Fréchet standardised dataset looks like



Application to pollution data - Pairwise modelling

Pairwise analyses

Using the regular MLE procedure we fit the Hüsler-Reiss and Extremal-t models.

Estimates of the exponent coefficient are provided in **blue**.

Pollutants	Hüsler-Reiss : $\hat{\lambda}/\widehat{\vartheta}_{\mathbf{HR}}$	Extremal-t : $(\hat{\rho},\hat{\nu})/\widehat{\vartheta}_{\mathbf{ET}}$
PM_{10} & NO	0.69/**1.51**	(0.84, 4.54)/**1.49**
PM_{10} & SO_2	0.92/**1.64**	(0.57, 2.08)/**1.58**
NO & SO_2	1.11/**1.73**	(0.41, 2.36)/**1.69**



How did we get those results?

We use routines from the package **`ExtremalDep`** as follows.

Application to pollution data - Pairwise modelling

```
## Focusing on PM10 and NO
dPN <- Leeds.frechet[apply(Leeds.frechet,1,function(x) !any(is.na(x))),1:2]

## Fiting the Husler-Reiss model
PN.hr <- fExtDep(method="Composite", data=dPN, model="HR", par.start=1)
PN.hr$par
```

```
## [1] 0.689
```

```
index.ExtDep(object="extremal", model="HR", par=PN.hr$par)
```

```
## [1] 1.509177
```

Application to pollution data - Pairwise modelling

```
## Fitting the Extremal-t model
PN.et <- fExtDep(method="Composite", data=dPN, model="ET", par.start=c(0.5,1))
PN.et$par
```

```
## [1] 0.839 4.538
```

```
index.ExtDep(object="extremal", model="ET", par=PN.et$par)
```

```
## [1] 1.485618
```

Application to pollution data - Pairwise modelling

```
## Focusing on PM10 and SO2
dPS <- Leeds.frechet[apply(Leeds.frechet,1,function(x) !any(is.na(x))),c(1,5)]

## Fiting the Husler-Reiss model
PS.hr <- fExtDep(method="Composite", data=dPS, model="HR", par.start=1)
PS.hr$par
```

```
## [1] 0.919
```

```
index.ExtDep(object="extremal", model="HR", par=PS.hr$par)
```

```
## [1] 1.641904
```

Application to pollution data - Pairwise modelling

```
## Fitting the Extremal-t model
PS.et <- fExtDep(method="Composite", data=dPS, model="ET", par.start=c(0.5,1))
PS.et$par
```

```
## [1] 0.567 2.084
```

```
index.ExtDep(object="extremal", model="ET", par=PS.et$par)
```

```
## [1] 1.577667
```

Application to pollution data - Pairwise modelling

```
## Focusing on NO and SO2
dNS <- Leeds.frechet[apply(Leeds.frechet,1,function(x) !any(is.na(x))),c(2,5)]

## Fiting the Husler-Reiss model
NS.hr <- fExtDep(method="Composite", data=dNS, model="HR", par.start=1)
NS.hr$par
```

```
## [1] 1.109
```

```
index.ExtDep(object="extremal", model="HR", par=NS.hr$par)
```

```
## [1] 1.73257
```

Application to pollution data - Pairwise modelling

```
## Fitting the Extremal-t model
NS.et <- fExtDep(method="Composite", data=dNS, model="ET", par.start=c(0.5,1))
NS.et$par
```

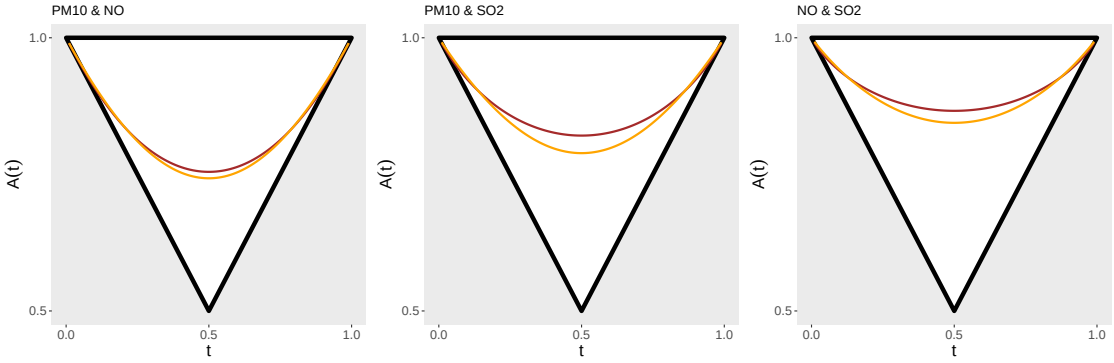
```
## [1] 0.407 2.359
```

```
index.ExtDep(object="extremal", model="ET", par=NS.et$par)
```

```
## [1] 1.688637
```

Application to pollution data - Visualising the pairwise dependence

We now display the estimated Pickands dependence function for both models (Hüsler-Reiss and **Extremal-t**)



Application to pollution data - Visualising the pairwise dependence

These plots are obtained by first creating a data.frame with evaluations of the Pickands function.

```
vals <- simplex(d=2, n=100)

Pick.dat <- data.frame(
  t = vals[,1],
  PN_hr = apply(vals, 1, index.ExtDep, object="pickands", model="HR", par=PN.hr$par),
  PN_et = apply(vals, 1, index.ExtDep, object="pickands", model="ET", par=PN.et$par),
  PS_hr = apply(vals, 1, index.ExtDep, object="pickands", model="HR", par=PS.hr$par),
  PS_et = apply(vals, 1, index.ExtDep, object="pickands", model="ET", par=PS.et$par),
  NS_hr = apply(vals, 1, index.ExtDep, object="pickands", model="HR", par=NS.hr$par),
  NS_et = apply(vals, 1, index.ExtDep, object="pickands", model="ET", par=NS.et$par))

Pick.dat <- Pick.dat[-c(1, 100),]
```

Application to pollution data - Visualising the pairwise dependence

To generate the plot for PM10 and NO:

```
df <- data.frame(x=c(0,0.5,1), y=c(1,0.5,1))

ggplot(data=Pick.dat, aes(x=t, y=PN_hr)) +
  geom_polygon(data = df, aes(x=x,y=y), fill="white", linewidth=2, col="black") +
  geom_line(data=Pick.dat, aes(x=t, y=PN_hr), col="brown", lwd=1, lty=1) +
  geom_line(data=Pick.dat, aes(x=t, y=PN_et), col="orange", lwd=1, lty=1) +
  scale_x_continuous(name=expression(t), limits=c(0, 1), breaks=c(0, 0.5, 1)) +
  scale_y_continuous(name=expression(A(t)), limits=c(0.5, 1), breaks=c(0.5, 1)) +
  ggtitle("PM10 & NO") +
  theme(panel.grid.major = element_blank(), panel.grid.minor = element_blank(),
        axis.text=element_text(size=12),
        axis.title=element_text(size=16,face="bold"))
```

Application to pollution data - Probability of joint events

We are interested in combinations of the events $PM_{10} > 125$, $NO > 359$ and $SO_2 > 125$, which correspond to the 92, 90 and 93% upper quantiles.

Event	Excess / n	Empirical Est.	HR Est.	ET Est.
$PM_{10} > 125$ and $NO > 359$	27/528	$0.051(0.032, 0.070)$	0.046	0.048
$PM_{10} > 125$ and $SO_2 > 125$	9/528	$0.017(0.008, 0.032)$	0.029	0.032
$NO > 359$ and $SO_2 > 125$	6/528	$0.011(0.004, 0.025)$	0.026	0.026

Application to pollution data - Probability of joint events

To reproduce this table, we need to write some functions. In particular the following calculates confidence intervals for empirical proportions.

```
conf.int <- function(exceed, n, alpha=0.05){
  p <- exceed/n
  if(p >= 0.02){# Normal approximation
    CIinf <- round( p - qnorm(1-alpha/2) * sqrt((p*(1-p))/n), 4)
    CIsup <- round( p + qnorm(1-alpha/2) * sqrt((p*(1-p))/n), 4)
  }else{# exact binomial
    CIinf <- 1 / (1 + (n-exceed+1) / exceed *
                  qf(1-alpha/2 ,df1=2*(n-exceed+1), df2=2*exceed))
    ci.qf <- qf(1-alpha/2, df1=2*(exceed+1), df2=2*(n-exceed))
    CIsup <- (exceed+1) / (n-exceed) * ci.qf / (1+(exceed+1)/(n-exceed)*ci.qf)
  }
  return(data.frame(est=p, CIinf = CIinf, CIsup= CIsup))
}
```

Application to pollution data - Probability of joint events

```
threshold_PM10 <- as.numeric(quantile(winterdat[,1],0.92, na.rm=TRUE))
threshold_PM10 # PM10 > 125
```

```
## [1] 125
```

```
threshold_NO <- as.numeric(quantile(winterdat[,2],0.90, na.rm=TRUE))
threshold_NO # NO > 359
```

```
## [1] 359.2
```

```
threshold_SO2 <- as.numeric(quantile(winterdat[,5],0.93, na.rm=TRUE))
threshold_SO2 # SO2 > 125
```

```
## [1] 124.6
```

Application to pollution data - Probability of joint events

Using the quantile probabilities, we derive the quantiles on the unit Fréchet scale.

```
# thresholds on Frechet scale
thresh <- as.numeric(c(quantile(Leeds.frechet[,1],0.92, na.rm=TRUE),
                       quantile(Leeds.frechet[,2],0.90, na.rm=TRUE),
                       quantile(Leeds.frechet[,5],0.93, na.rm=TRUE)))
```

These will be used to compute the empirical probabilities of excess as follows:

$$\mathbb{P}(X > x, Y > y) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{x_i > x, y_i > y\},$$

where $x_i, i = 1, \ldots, n$ and $y_i, i = 1, \ldots, n$ are realisations of x and Y.

Application to pollution data - Probability of joint events

Using the results from the previous slide, we have

```
## Focusing on PM10 and NO
excess_PN <- sum((Leeds.frechet[,1]>thresh[1]) *
                   (Leeds.frechet[,2]>thresh[2]), na.rm=T)
conf.int(excess_PN, nrow(dPN))
```

```
##          est  CIinf  CIsup
## 1 0.05113636 0.0323 0.0699
```

Application to pollution data - Probability of joint events

Let $G_{\mathrm{m}}(\cdot, \theta_{\mathrm{m}})$ represent the distribution function from a model m, with marginals given by $G(\cdot)$. The modelled probabilities of excess are given by

$$\mathbb{P}(X > x, Y > y) = G(x) + G(y) - G_{\mathrm{m}}((x, y), \theta_{\mathrm{m}})$$

```
pExtDep(thresh[1:2], model="HR", par=PN.hr$par, type="upper")
```

```
## [1] 0.04646109
```

```
pExtDep(thresh[1:2], model="ET", par=PN.et$par, type="upper")
```

```
## [1] 0.0475251
```

Application to pollution data - Probability of joint events

For the other two pairs...

```
## Focusing on PM10 and SO2
excess_PS <- sum((Leeds.frechet[,1]>thresh[1]) *
                   (Leeds.frechet[,5]>thresh[3]), na.rm=T)
conf.int(excess_PS, nrow(dPS))
```

```
##          est       CIinf      CIsup
## 1 0.01704545 0.007823139 0.03211014
```

```
pExtDep(thresh[c(1,3)], model="HR", par=PS.hr$par, type="upper")
```

```
## [1] 0.02897947
```

```
pExtDep(thresh[c(1,3)], model="ET", par=PS.et$par, type="upper")
```

```
## [1] 0.03196872
```

Application to pollution data - Probability of joint events

```
## Focusing on NO and SO2
excess_NS <- sum((Leeds.frechet[,2]>thresh[2]) *
                   (Leeds.frechet[,5]>thresh[3]), na.rm=T)
conf.int(excess_NS, nrow(dNS))
```

```
##          est       CIinf      CIsup
## 1 0.01136364 0.004181356 0.02456881
```

```
pExtDep(thresh[2:3], model="HR", par=NS.hr$par, type="upper")
```

```
## [1] 0.02633707
```

```
pExtDep(thresh[2:3], model="ET", par=NS.et$par, type="upper")
```

```
## [1] 0.02649148
```

Application to pollution data - Return levels

Note that we have been using **standardised data** (on unit Fréchet scale) to fit the models but the **return levels should be on the original data scale**.

The transformation is not available here so we will provide the return levels on the unit Fréchet scale.

The original analysis of this data consider transforming these data to unit Fréchet by fitting a GPD distribution to the exceedances of the 70% empirical quantile and using the empirical distribution function for the observations below the threshold.

Application to pollution data - Return levels

1. Create grid of values to evaluate the return levels

```
# Let's get an idea of what value each pollutant can take
apply(Leeds.frechet,2,range, na.rm=TRUE)
```

```
##              PM10          NO          NO2           O3          SO2
## [1,]   0.1586642   0.1679512    0.1572432 1.763705e-01    0.1771894
## [2,] 847.8055120 809.2351597 1202.4048387 1.079546e+04 2059.8140405
```

```
len <- 50
PM10.frech <- seq(from=1, to=900, length=len)
NO.frech <- seq(from=1, to=900, length=len)
SO2.frech <- seq(from=1, to=2100, length=len)

PN.frech <- expand.grid(PM10.frech, NO.frech)
PS.frech <- expand.grid(PM10.frech, SO2.frech)
NS.frech <- expand.grid(NO.frech, SO2.frech)
```

2. Evaluate the return levels

First we focus on the PM10 & NO dataset.

```
# For Husler-Reiss model fit
rl.PN.hr <- apply(PN.frech, 1, pExtDep, model="HR", par=PN.hr$par, type="upper")
rl.PN.hr <- matrix(rl.PN.hr, nrow=len, ncol=len, byrow=TRUE)

# For Extremal-t model fit
rl.PN.et <- apply(PN.frech, 1, pExtDep, model="ET", par=PN.et$par, type="upper")
rl.PN.et <- matrix(rl.PN.et, nrow=len, ncol=len, byrow=TRUE)
```

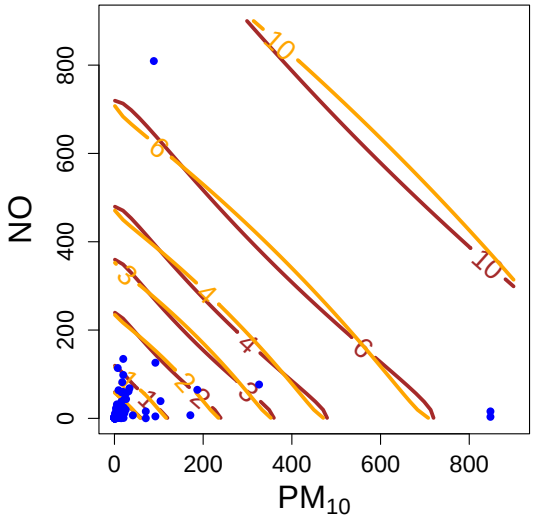
4. Make the plot(s)!

```
rp <-c(0.5,1,2,3,4,6,10)

par(mar=c(4.3, 4.6, 0.5, 0.1))
plot(-100,-100, xlab=expression(PM[10]), ylab="NO", cex.lab=1.8,
     cex.axis=1.3, xlim=range(PM10.frech), ylim=range(NO.frech) )
contour(PM10.frech, NO.frech, (1/rl.PN.hr)/120, levels=rp, add=TRUE,
        labcex=1.6, lwd=3.4, col="brown")
contour(PM10.frech, NO.frech, (1/rl.PN.et)/120, levels=rp, add=TRUE,
        labcex=1.6, lwd=3.4, col="orange")
points(Leeds.frechet[,1:2], pch=16, col="blue")
```

Application to pollution data - Return levels

4. Make the plot(s)!



Application to pollution data - Return levels

Replicate for the PM10 & SO2 pair...

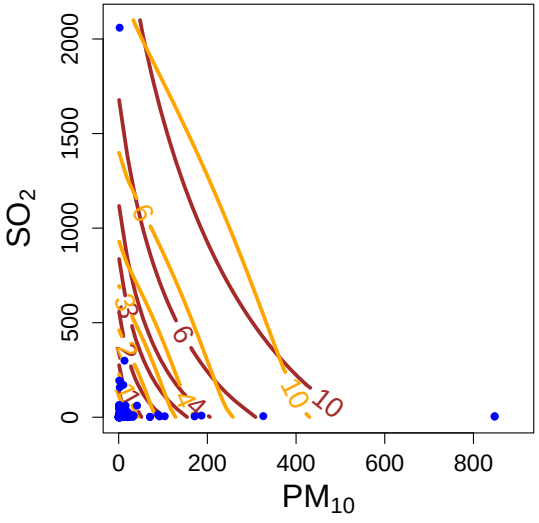
```
# For Husler-Reiss model fit
rl.PS.hr <- apply(PS.frech, 1, pExtDep, model="HR", par=PS.hr$par, type="upper")
rl.PS.hr <- matrix(rl.PS.hr, nrow=len, ncol=len, byrow=TRUE)

# For Extremal-t model fit
rl.PS.et <- apply(PS.frech, 1, pExtDep, model="ET", par=PS.et$par, type="upper")
rl.PS.et <- matrix(rl.PS.et, nrow=len, ncol=len, byrow=TRUE)
```

Application to pollution data - Return levels

```
par(mar=c(4.3, 4.6, 0.5, 0.1))
plot(-100,-100, xlab=expression(PM[10]), ylab=expression(SO[2]), cex.lab=1
     cex.axis=1.3, xlim=range(PM10.frech), ylim=range(SO2.frech) )
contour(PM10.frech, SO2.frech, (1/rl.PS.hr)/120, levels=rp, add=TRUE,
        labcex=1.6, lwd=3.4, col="brown")
contour(PM10.frech, SO2.frech, (1/rl.PS.et)/120, levels=rp, add=TRUE,
        labcex=1.6, lwd=3.4, col="orange")
points(Leeds.frechet[,c(1,5)], pch=16, col="blue")
```

Application to pollution data - Return levels



Application to pollution data - Return levels

Replicate for the NO & SO2 pair...

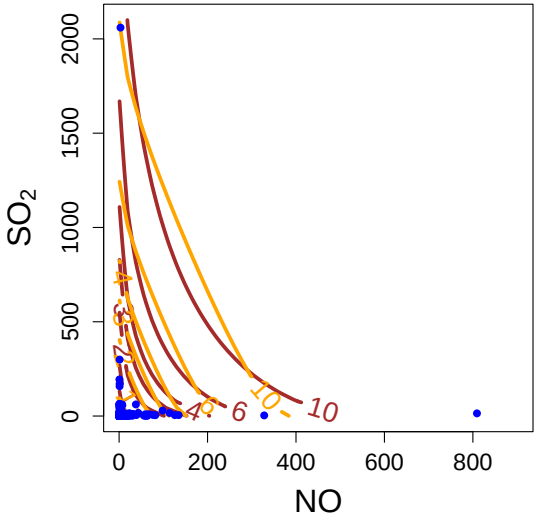
```
# For Husler-Reiss model fit
rl.NS.hr <- apply(NS.frech, 1, pExtDep, model="HR", par=NS.hr$par, type="upper")
rl.NS.hr <- matrix(rl.NS.hr, nrow=len, ncol=len, byrow=TRUE)

# For Extremal-t model fit
rl.NS.et <- apply(NS.frech, 1, pExtDep, model="ET", par=NS.et$par, type="upper")
rl.NS.et <- matrix(rl.NS.et, nrow=len, ncol=len, byrow=TRUE)
```

Application to pollution data - Return levels

```
par(mar=c(4.3, 4.6, 0.5, 0.1))
plot(-100,-100, xlab="NO", ylab=expression(SO[2]), cex.lab=1.8,
     cex.axis=1.3, xlim=range(NO.frech), ylim=range(SO2.frech) )
contour(NO.frech, SO2.frech, (1/rl.NS.hr)/120, levels=rp, add=TRUE,
        labcex=1.6, lwd=3.4, col="brown")
contour(NO.frech, SO2.frech, (1/rl.NS.et)/120, levels=rp, add=TRUE,
        labcex=1.6, lwd=3.4, col="orange")
points(Leeds.frechet[,c(2,5)], pch=16, col="blue")
```

Application to pollution data - Return levels



7. Application to wind speed data



Application to wind speed data

Data

- Type: Weekly maximum wind speed (km/h)
- Locations: 4 monitoring stations across Oklahoma, USA.
- Location names: CLOU, CLAY, PAUL & SALL.
- Period: March-May period during 1996-2012.

The observations are **marginally pre-transformed to unit Fréchet distributions**.

Application to wind speed data - Model fits

Use pairwise composite likelihood to fit the Hüsler-Reiss and Extremal-t models.

Stations	Hüsler-Reiss Estimates	Extremal-t estimates
CLOU CLAY SALL	$(0.85, 1.01, 0.91)$	$(0.67, 0.57, 0.69, 2.89)$
CLOU CLAY PAUL	$(0.87, 1.10, 0.92)$	$(0.59, 0.50, 0.69, 2.53)$
CLAY SALL PAUL	$(0.77, 0.87, 0.88)$	$(0.65, 0.61, 0.53, 1.55)$
CLOU SALL PAUL	$(1.02, 1.10, 1.05)$	$(0.37, 0.40, 0.42, 1.88)$

```
data(Wind) # available in ExtremalDep package


extRemes::fevd(CLOU.CLAY.SALL[,1])$results$par # unit Frechet margins!
```

```
## location    scale    shape
## 1.004637 1.022953 1.048417
```

```
# fGEV(CLOU.CLAY.SALL[,1])$est
```

Application to wind speed data - Model fits

```
# Starting values
p0.hr <- c(rep(1,3))
p0.et <- c(rep(0.5,3),1)

# CLOU CLAY SALL Stations
hr1 <- fExtDep(method="Composite", CLOU.CLAY.SALL, model="HR", p0.hr)
hr1$par
```

```
## [1] 0.854 1.007 0.911
```

```
et1 <- fExtDep(method="Composite", CLOU.CLAY.SALL, model="ET", p0.et)
et1$par
```

```
## [1] 0.666 0.569 0.686 2.898
```

Application to wind speed data - Model fits

```
# CLOU CLAY PAUL Stations
hr2 <- fExtDep(method="Composite", CLOU.CLAY.PAUL, model="HR", p0.hr)
hr2$par
```

```
## [1] 0.870 1.104 0.916
```

```
et2 <- fExtDep(method="Composite", CLOU.CLAY.PAUL, model="ET", p0.et)
et2$par
```

```
## [1] 0.594 0.498 0.692 2.527
```

Application to wind speed data - Model fits

```
# CLAY SALL PAUL Stations
hr3 <- fExtDep(method="Composite", CLAY.SALL.PAUL, model="HR", p0.hr)
hr3$par
```

```
## [1] 0.774 0.870 0.877
```

```
et3 <- fExtDep(method="Composite", CLAY.SALL.PAUL, model="ET", p0.et)
et3$par
```

```
## [1] 0.649 0.610 0.531 1.547
```

Application to wind speed data - Model fits

```
# CLOU SALL PAUL Stations
hr4 <- fExtDep(method="Composite", CLOU.SALL.PAUL, model="HR", p0.hr)
hr4$par
```

```
## [1] 1.026 1.099 1.049
```

```
et4 <- fExtDep(method="Composite", CLOU.SALL.PAUL, model="ET", p0.et)
et4$par
```

```
## [1] 0.372 0.400 0.417 1.876
```

Application to wind speed data - Dependence summaries

The estimated extremal and upper tail dependence coefficients for the Hüsler-Reiss and Extremal-t models (resp. $\hat{\vartheta}_{HR}$, $\hat{\chi}_{HR}$, $\hat{\vartheta}_{ET}$, $\hat{\chi}_{ET}$) are:

Stations	$\hat{\vartheta}_{HR}$	$\hat{\chi}_{HR}$	$\hat{\vartheta}_{ET}$	$\hat{\chi}_{ET}$
CLOU CLAY SALL	2.150	0.220	2.019	0.250
CLOU CLAY PAUL	2.187	0.200	2.020	0.243
CLAY SALL PAUL	2.058	0.262	1.850	0.324
CLOU SALL PAUL	2.294	0.165	2.134	0.182

The fits of the Hüsler-Reiss model suggests a stronger independence behaviour in the extremes than the Extremal-t ($\hat{\vartheta}$ close to 3 and $\hat{\chi}$ closer to zero).

Application to wind speed data - Dependence summaries

```
# CLOU CLAY SALL stations
V <- index.ExtDep(object="extremal", model="HR", par=hr1$par)
V1 <- index.ExtDep(object="extremal", model="HR", par=hr1$par[1])
V2 <- index.ExtDep(object="extremal", model="HR", par=hr1$par[2])
V3 <- index.ExtDep(object="extremal", model="HR", par=hr1$par[3])
c(V, 3-V1-V2-V3+V) # inclusion-exclusion principle
```

```
## [1] 2.1502920 0.2196272
```

```
v <- index.ExtDep(object="extremal", model="ET", par=et1$par)
v1 <- index.ExtDep(object="extremal", model="ET", par=et1$par[c(1,4)])
v2 <- index.ExtDep(object="extremal", model="ET", par=et1$par[c(2,4)])
v3 <- index.ExtDep(object="extremal", model="ET", par=et1$par[c(3,4)])
c(v, 3-v1-v2-v3+v) # inclusion-exclusion principle
```

```
## [1] 2.0183005 0.2501648
```

Application to wind speed data - Dependence summaries

```
# CLOU CLAY PAUL stations
V <- index.ExtDep(object="extremal", model="HR", par=hr2$par)
V1 <- index.ExtDep(object="extremal", model="HR", par=hr2$par[1])
V2 <- index.ExtDep(object="extremal", model="HR", par=hr2$par[2])
V3 <- index.ExtDep(object="extremal", model="HR", par=hr2$par[3])
c(V, 3-V1-V2-V3+V) # inclusion-exclusion principle
```

```
## [1] 2.1866869 0.2002474
```

```
v <- index.ExtDep(object="extremal", model="ET", par=et2$par)
v1 <- index.ExtDep(object="extremal", model="ET", par=et2$par[c(1,4)])
v2 <- index.ExtDep(object="extremal", model="ET", par=et2$par[c(2,4)])
v3 <- index.ExtDep(object="extremal", model="ET", par=et2$par[c(3,4)])
c(v, 3-v1-v2-v3+v) # inclusion-exclusion principle
```

```
## [1] 2.020173 0.242777
```

Application to wind speed data - Dependence summaries

```
# CLAY SALL PAUL stations
V <- index.ExtDep(object="extremal", model="HR", par=hr3$par)
V1 <- index.ExtDep(object="extremal", model="HR", par=hr3$par[1])
V2 <- index.ExtDep(object="extremal", model="HR", par=hr3$par[2])
V3 <- index.ExtDep(object="extremal", model="HR", par=hr3$par[3])
c(V, 3-V1-V2-V3+V) # inclusion-exclusion principle
```

```
## [1] 2.0578050 0.2615228
```

```
v <- index.ExtDep(object="extremal", model="ET", par=et3$par)
v1 <- index.ExtDep(object="extremal", model="ET", par=et3$par[c(1,4)])
v2 <- index.ExtDep(object="extremal", model="ET", par=et3$par[c(2,4)])
v3 <- index.ExtDep(object="extremal", model="ET", par=et3$par[c(3,4)])
c(v, 3-v1-v2-v3+v) # inclusion-exclusion principle
```

```
## [1] 1.8498073 0.3243536
```

Application to wind speed data - Dependence summaries

```
# CLOU SALL PAUL stations
V <- index.ExtDep(object="extremal", model="HR", par=hr4$par)
V1 <- index.ExtDep(object="extremal", model="HR", par=hr4$par[1])
V2 <- index.ExtDep(object="extremal", model="HR", par=hr4$par[2])
V3 <- index.ExtDep(object="extremal", model="HR", par=hr4$par[3])
c(V, 3-V1-V2-V3+V) # inclusion-exclusion principle
```

```
## [1] 2.2939268 0.1647645
```

```
v <- index.ExtDep(object="extremal", model="ET", par=et4$par)
v1 <- index.ExtDep(object="extremal", model="ET", par=et4$par[c(1,4)])
v2 <- index.ExtDep(object="extremal", model="ET", par=et4$par[c(2,4)])
v3 <- index.ExtDep(object="extremal", model="ET", par=et4$par[c(3,4)])
c(v, 3-v1-v2-v3+v) # inclusion-exclusion principle
```

```
## [1] 2.1337858 0.1818196
```

Application to wind speed data - Exceedance probabilities

Using the **tail dependence function** we are able to compute some conditional probability of exceeding some high values.

The 70% and 90% **wind speed quantiles** at CLOU, CLAY and SALL are respectively $(17.98, 20.27, 24.36)$ and $(22.17, 24.18, 29.42)$. On the unit Fréchet scale these quantiles are $(2.60, 2.96, 2.87)$ and $(11.77, 9.85, 9.97)$.

We want to estimate the probabilities

$$\mathbb{P}\left(\mathbf{CLOU} > 22.17, \mathbf{CLAY} > 24.18 | \mathbf{SALL} > 24.36\right) \tag{1}$$

and

$$\mathbb{P}\left(\mathbf{CLOU} > 22.17 | \mathbf{CLAY} > 20.27, \mathbf{SALL} > 24.36\right). \tag{2}$$

Application to wind speed data - Exceedance probabilities

We have

$$(1) = \frac{\mathbb{P}\left(\text{CLOU} > 22.17, \text{CLAY} > 24.18, \text{SALL} > 24.36\right)}{\mathbb{P}\left(\text{SALL} > 24.36\right)}$$
$$\approx \frac{R(11.77, 9.85, 2.87)}{R(2.87)} := p$$

We consider the estimator:

$$\hat{p} = \frac{R(11.77, 9.85, 2.87; \hat{\boldsymbol{\theta}})}{R(2.87; \hat{\boldsymbol{\theta}})},$$

and obtain $\hat{p}_{HR} = 0.1116$ and $\hat{p}_{ET} = 0.1196$. The Extremal-t model allows for a greater probability of extreme winds at CLOU and CLAY given that the wind is already moderately large at SALL.

Application to wind speed data - Exceedance probabilities

```
data <- CLOU.CLAY.SALL
n <-nrow(data)

q70 <- as.numeric(apply(data, 2, quantile, 0.70))
q90 <- as.numeric(apply(data, 2, quantile, 0.90))

pExtDep(q=c(q90[1:2], q70[3]), type="upper", model="HR", par=hr1$par) /
  pGEV(q70[3], loc=1, scale=1, shape=1, lower.tail=FALSE)
```

```
## [1] 0.1113398
```

```
pExtDep(q=c(q90[1:2], q70[3]), type="upper", model="ET", par=et1$par) /
  pGEV(q70[3], loc=1, scale=1, shape=1, lower.tail=FALSE)
```

```
## [1] 0.1185973
```

Application to wind speed data - Exceedance probabilities

Note that the empirical estimate is $\hat{p}_{emp} = 0.0781$, which is obtained by computing

$$\hat{p} = \frac{\sum_{i=1}^{n} \mathbb{I}\{\mathrm{CLOU}_i > 11.77, \mathrm{CLAY}_i > 9.85, \mathrm{SALL}_i > 2.87\}}{\sum_{i=1}^{n} \mathbb{I}\{\mathrm{SALL}_i > 2.87\}},$$

where $\mathrm{CLOU}_i, \mathrm{CLAY}_i$ and SALL_i indicate observations on the unit Fréchet scale.

```
# Empirically:
conditionalData <- data[data[,3] > q70[3],]
exceed <- sum( apply(conditionalData[,1:2],1, function(x) all(x> q90[1:2])) )
exceed/nrow(conditionalData)
```

```
## [1] 0.078125
```

Application to wind speed data - Exceedance probabilities

We also have

$$(2) = \frac{\mathbb{P}(\text{CLOU} > 22.17, \text{CLAY} > 20.27, \text{SALL} > 24.36)}{\mathbb{P}(\text{CLAY} > 20.27, \text{SALL} > 24.36)} \approx \frac{R(11.77, 2, 96, 2.87)}{R(2.96, 2.87)} := p$$

We consider the estimator:

$$\hat{p} = \frac{R(11.77, 2.96, 2.87; \hat{\boldsymbol{\theta}})}{R(2.96, 2.87; \hat{\boldsymbol{\theta}})},$$

and obtain $\hat{p}_{HR} = 0.3138$ and $\hat{p}_{ET} = 0.3207$. The Hüsler-Reiss model allows for a much lower probability of extreme winds at CLOU given that the wind is already moderately large at CLAY and SALL. Note that the empirical estimate is $\hat{p}_{emp} = 0.3784$.

Application to wind speed data - Exceedance probabilities

```
pExtDep(q=c(q90[1], q70[2:3]), type="upper", model="HR", par=hr1$par) /
  pExtDep(q70[2:3], type="upper", model="HR", par=hr1$par[3])
```

```
## [1] 0.3138065
```

```
pExtDep(q=c(q90[1], q70[2:3]), type="upper", model="ET", par=et1$par) /
  pExtDep(q70[2:3], type="upper", model="ET", par=et1$par[3:4])
```

```
## [1] 0.3207207
```

Application to wind speed data - Exceedance probabilities

Note that the empirical estimate is $\hat{p}_{emp} = 0.1351$, which is obtained by computing

$$\hat{p} = \frac{\sum_{i=1}^{n} \mathbb{I}\{\mathrm{CLOU}_i > 11.77, \mathrm{CLAY}_i > 2.96, \mathrm{SALL}_i > 2.87\}}{\sum_{i=1}^{n} \mathbb{I}\{\mathrm{CLAY}_i > 2.96, \mathrm{SALL}_i > 2.87\}},$$

where $\mathrm{CLOU}_i, \mathrm{CLAY}_i$ and SALL_i indicate observations on the unit Fréchet scale.

```
# Empirically:
conditionalData <- data[apply(data[,2:3],1, function(x) all(x> q70[2:3])),]
exceed <- sum( conditionalData[,1]> q90[1])
exceed/nrow(conditionalData)
```

```
## [1] 0.3783784
```

References



Sources and recommending reading

1. **[BGST04]** Beirlant, J., Goegebeur, Y., Segers, J. & Teugels J. (2004) *Statistics of Extremes: Theory and Applications*. John Wiley & Sons. Chapter 8 & 9

2. **[C01]** Coles, S. (2001) An Introduction to Statistical Modeling of Extreme Values. Springer. Chapter 7.