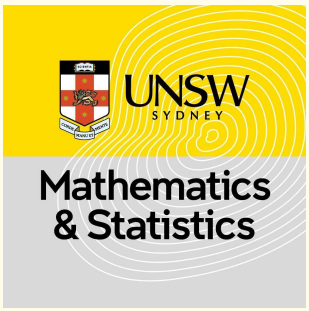


MATH5805
A Modern Introduction to Extreme Value Theory
Term 1, 2025

Dr. Boris Béranger

Week 7 - Tutorial Solutions

Exercise 1

Let E_d be a d-dimensional Euclidean space and suppose that H is a non-differentiable spectral measure which places mass $1/d$ on the standard Euclidean basis vectors $\boldsymbol{e}_k$, for $k = 1, \ldots, d$. Let G be a d-dimensional multivariate distribution with unit Fréchet margins and let V be the associated exponent function. For $d = 3$, show that

$$V(\boldsymbol{z}) = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3},$$

where $\boldsymbol{z}$ is a positive vector. Hence show that G has independent margins. Try to generalise the result to any natural d.

Solution:
Let $\mathbb{W}_3 = \{(a, b, 1-a-b) \in [0,1]^3 : s + t \leq 1\}$ denote the 2-simplex — the k-simplex defines the convex combination of $k+1$ vertices. Let H be the spectral measure which places mass d^{-1} on the vertices of $\mathbb{W}$, and zero everywhere else. Then

$$\begin{aligned} V(x,y,z) &= 3 \int_{\mathbb{W}} \max\left(\frac{a}{x}, \frac{b}{y}, \frac{c}{z}\right) H(d\boldsymbol{\omega}) \\ &= 3\left[\frac{1}{3}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)\right] \\ &= \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \end{aligned}$$

since, for example, $\max(ax^{-1}, by^{-1}, cz^{-1}) = x^{-1}$ at $H(\boldsymbol{e}_1) = 1/3$. We only need to consider the points $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ since $H(\boldsymbol{\omega}) = 0$ everywhere else. We then have that

$$G(x,y,z) = \exp\left(-V(x,y,z)\right) = \mathrm{e}^{-x}\mathrm{e}^{-y}\mathrm{e}^{-z},$$

which implies that the margins are independent since the joint distribution function is the product of its margins.

Exercise 2

Repeat the previous exercise for the case in which H is a non-differentiable spectral measure which places mass 1 on $\boldsymbol{\omega} = (d^{-1}, \ldots, d^{-1})$. Compute the multivariate distribution G and show that the margins are perfectly positively correlated.

Solution:
We have that

$$\begin{aligned} V(z_1,\ldots,z_d) &= d\int_{\mathbb{W}_d} \max_{k=1,\ldots,d}\left\{\frac{\omega_k}{z_d}\right\} H(\mathrm{d}\boldsymbol{\omega}) \\ &= d\max\left\{\frac{d^{-1}}{z_1},\ldots,\frac{d^{-1}}{z_d}\right\} \\ &= dd^{-1}\max\left\{\frac{1}{z_1},\ldots,\frac{1}{z_d}\right\} \\ &= \max\left\{\frac{1}{z_1},\ldots,\frac{1}{z_d}\right\}. \end{aligned}$$

It follows that

$$G(z_1,\ldots,z_d) = \exp\left(-\max\left\{z_1^{-1},\ldots,z_d^{-1}\right\}\right).$$

Heuristically, it is clear to see that each of the marginal distributions will be determined by the smallest z_k, for $k=1,\ldots,d$. It follows that all the margins are perfectly correlated since the $\boldsymbol{z}$ can be chosen to be any positive vector.

Exercise 3

Show that the exponent function $V:\mathbb{R}_+^d \to (0,\infty)$ satisfies the following properties

a) $V(a\boldsymbol{z}) = a^{-1}V(\boldsymbol{z})$, for positive a.

Solution:
We have that

$$V(a\boldsymbol{z}) = d\int_{\mathbb{W}} \max_{k=1,\ldots,d}\left\{\frac{\omega_k}{az_k}\right\} H(\mathrm{d}\boldsymbol{\omega}) = a^{-1}d\int_{\mathbb{W}} \max_{k=1,\ldots,d}\left\{\frac{\omega_k}{z_k}\right\} H(\mathrm{d}\boldsymbol{\omega}) = a^{-1}V(\boldsymbol{z}).$$

b) $V(a\boldsymbol{z}+(1-a)\boldsymbol{z}') \le aV(\boldsymbol{z})+(1-a)V(\boldsymbol{z}')$, for $0\le a\le 1$ and positive $\boldsymbol{z}$ and $\boldsymbol{z}'$.

Solution:
The result follows from convexity of the max function and linearity of the integration operator. The convexity of the max function is demonstrated as $\max(ax+(1-a)y, ax'+(1-a)y') \le \theta\max(x,x')+(1-\theta)\max(y,y')$ for $0\le a\le 1$.

c) $\max\left(z_1^{-1},\ldots,z_d^{-1}\right) \le V(\boldsymbol{z}) \le z_1^{-1}+\cdots+z_d^{-1}$, for positive z_k, for $k=1,\ldots,d$.

Solution:
Let

$$A_{\boldsymbol{z}} := \left\{\boldsymbol{v}\in\mathbb{R}_+^d : v_1 > z_1, \text{ or } v_2 > z_2, \cdots, \text{ or } v_d > z_d\right\}$$

be the extremal set $(\boldsymbol{0},\boldsymbol{\infty})\setminus(\boldsymbol{0},\boldsymbol{z})$. Then $V(\boldsymbol{z}) \equiv \eta A_{\boldsymbol{z}}$. Consider the sets

$$A_{z_k} := \left\{\boldsymbol{v}\in\mathbb{R}_+^d : v_k > z_k\right\},$$

for $k=1,\ldots,d$. Observe that

$$V(\infty,\ldots,\infty,z_k,\infty,\ldots,\infty) = d\int_{\mathbb{W}_d}\frac{\omega_k}{z_k}H(\mathrm{d}\boldsymbol{\omega}) = \frac{1}{z_k} = \eta A_{z_k}.$$

Then we have that

$$V(\boldsymbol{z}) = \eta A_{\boldsymbol{z}} = \eta \bigcup_{k=1}^{d} A_{z_k} \leq \sum_{k=1}^{d} \eta A_{z_k} = \sum_{k=1}^{d} V(\infty, \ldots, \infty, z_k, \infty, \ldots, \infty) = \sum_{k=1}^{d} z_k^{-1},$$

by sub-additivity of measures. Additionally, we have the fact that

$$\eta A_{\boldsymbol{z}} = \eta \bigcup_{k=1}^{d} A_{z_k} \geq \max_{k=1,\ldots,d} \eta A_{z_k},$$

which essentially states the volume of the entire set is larger than the volume of the greatest component part. Using notation from above, we have that

$$V(\boldsymbol{z}) = \eta A_{\boldsymbol{z}} \geq \max_{k=1,\ldots,d} \eta A_{z_k} = \max_{k=1,\ldots,} \left\{ \frac{1}{z_k} \right\}.$$

Exercise 4

Show that the Pickands dependence function is convex.

Solution:
We shall consider dimension $d = 2$ for simplicity. Recall that we may write

$$\begin{aligned} A(t) &= \frac{z_1 z_2}{z_1 + z_2} V(z_1, z_2) \\ &= 2 \frac{z_1 z_2}{z_1 + z_2} \int_0^1 \frac{\omega}{z_1} \vee \frac{1-\omega}{z_2} \, H(\mathrm{d}\omega) \\ &= 2 \int_0^1 \omega \frac{z_2}{z_1 + z_2} \vee (1-\omega) \frac{z_1}{z_1 + z_2} \, H(\mathrm{d}\omega) \\ &= 2 \int_0^1 w(1-t) \vee (1-\omega) t \, H(\mathrm{d}\omega), \end{aligned}$$

where $t = z_1(z_1 + z_2)^{-1}$ and we have used the notation $a \vee b \equiv \max(a, b)$. The result then follows from the convexity of the max function and linearity of the integral operator.

Exercise 5

Show that the extremal coefficient $\vartheta \in [1, d]$ defines the number of independent margins of the random vector $\boldsymbol{Y}$. Consider the specific case where $d = 3$.

Solution:
It is clear from the bounds of Pickand's dependence function that $1 \leq \vartheta \leq d$. Recall that

$$V(\boldsymbol{z}) = A(s,t) \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right).$$

We have that $\vartheta = V(1,1,1) = 3A(1/3, 1/3)$ and for $z_1 = z_2 = z_3 = z$, $s = t = 1/3$. Therefore, this implies that

$$\begin{aligned} G(\boldsymbol{z}) &= \exp\left(-A(s,t) \left[\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right] \right) \\ &= \exp\left(\vartheta \log\left[G_1(z) \right] \right) \\ &= \exp\left(\log\left[G_1(z)^{\vartheta} \right] \right) \\ &= G_1(z)^{\vartheta}, \end{aligned}$$

where G_1 is the unit Fréchet distribution. If all components are independent, we have $G(\boldsymbol{z}) = G_1(z)^d$ while if they are compppletely dependent (i.e., $X_1 = X_2 = \cdots = X_d$), then $G(\boldsymbol{z}) = G_1(z)$