MATH5805
A Modern Introduction to Extreme Value Theory
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Week 5 - Tutorial Solutions

Exercise 1

Let F be a distribution function whose maxima converges to the Weibull distribution. This implies that F has finite right end-point and has maxima which converges to the GEV.

a) Let F_u be the excess distribution function. Explain why we cannot define F_u for $u > x^*$.

Solution:
Recall that $F_u(x) = \mathbb{P}(X - u \leq y | X > u)$. However, the event $\{X > x^*\}$ has null probability, and hence cannot be conditioned on.

b) Show that any positive scaling of F yields the same convergence to the GPD. That is, if F is the distribution function for a bounded random variable X, then $Y = kX$ also satisfies GPD convergence, for positive k.

Solution:
We have that

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}\left(X \leq \frac{y}{k}\right) = F\left(\frac{y}{k}\right).$$

Suppose that F is in the maximal domain of attraction of the GEV distribution. Then taking $v = ku$ satisfies convergence to the GPD for Y using the sketch proof provided in Week 4 Lecture slides, pp.18–19. This implies that convergence at x^* for X is equivalent to sufficiently large u for Y.

Exercise 2

Let $X_1, \ldots, X_n$ be an independent sequence of identically distributed random variables taking values in $(0,1) \times \mathbb{R}$. Let N be a point process with intensity measure Λ and let A be some measurable subset of $(0,1) \times \mathbb{R}$. Show that $N(A) \sim \text{Binomial}(n, \mathbb{P}(A))$, that is, the distribution of $N(A)$ is Binomially distributed with parameters n and $\mathbb{P}(A)$. Also compute the intensity measure $\Lambda(A)$.

Solution:
Let

$$\delta_{X_i}(A) = \begin{cases} 0, & \text{if } X_i \notin A \\ 1, & \text{if } X_i \in A \end{cases}$$

be the Dirac measure centred at X_i. Then $N = \sum_{i=1}^n \delta_{X_i}$. Each $\delta_{X_i}(A)$ is identical to a Bernoulli random variable with parameter $p_i = \mathbb{E}[\delta_{X_i}] = \mathbb{P}(A)$. Hence $N(A)$ is Binomially distributed with

parameters n and $\mathbb{P}(A)$ since it is obtained from the sum of independent and identically distributed Bernoulli random variables. The intensity measure is

$$\Lambda(A) = \mathbb{E}[N(A)] = \sum_{i=1}^{n} \mathbb{E}\left[\delta_{X_i}(A)\right] = n\mathbb{P}(A).$$

Exercise 3

Let N be a non-homogenous Poisson process on $\mathbb{R}^2$, with intensity measure

$$\Lambda(A) = \int_A \lambda(\mathbf{s})\,\mathrm{d}\mathbf{s} = \int_a^b \int_c^d \lambda(s)\,\mathrm{d}s\,\mathrm{d}t,$$

where $A = (a, b) \times (c, d)$.

Express N explicitly for the cases where

a) $A \subseteq (0, 1) \times \mathbb{R}_+$ and $\lambda(s) = \alpha s^{-\alpha-1}$, for positive α.

Solution:
Let $0 < a \leq b < 1$ and $0 < c \leq d < \infty$. Then

$$\begin{aligned}
\Lambda(A) &= \int_a^b \int_c^d \lambda(s)\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_a^b \int_c^d \alpha s^{-\alpha-1}\,\mathrm{d}s\,\mathrm{d}t \\
&= \left[-s^{-\alpha}\right]_c^b \int_a^b \mathrm{d}t \\
&= (b-a)(c^{-\alpha} - b^{-\alpha}).
\end{aligned}$$

Observe that the intensity is bounded for every construction of A. Then

$$\begin{aligned}
\mathbb{P}\left(N(A) = k\right) &= \exp\left(-\Lambda(A)\right) \frac{[\Lambda(A)]^k}{k!} \\
&= \exp\left(-(b-a)\left[c^{-\alpha} - d^{-\alpha}\right]\right) \frac{(b-a)^k \left(c^{-\alpha} - d^{-\alpha}\right)^k}{k!}.
\end{aligned}$$

b) $A \subseteq (0, 1) \times \mathbb{R}$ and $\lambda(s) = \mathrm{e}^{-s}$.

Solution:
Let $0 < a \leq b < 1$ and $c \leq d$. Then

$$\begin{aligned}
\Lambda(A) &= \int_a^b \int_c^d \lambda(s)\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_a^b \int_c^d \mathrm{e}^{-s}\,\mathrm{d}s\,\mathrm{d}t \\
&= (b-a)\left[-\mathrm{e}^{-s}\right]_c^d \\
&= (b-a)\left(\mathrm{e}^{-c} - \mathrm{e}^{-d}\right),
\end{aligned}$$

which is finite for every construction of A. Then

$$\mathbb{P}\left(N(A)=k\right)=\exp\left(-(b-a)\left[\mathrm{e}^{-c}-\mathrm{e}^{-d}\right]\right)\frac{(b-a)^k\left[\mathrm{e}^{-c}-\mathrm{e}^{-d}\right]^k}{k!}.$$

Exercise 4 (Difficult)

Show that if $F\in\mathcal{D}(\Lambda)$, then

$$\lim_{t\to x^*}\frac{\lim_{s\uparrow t}1-F(s)}{\lim_{s\downarrow t}1-F(s)}=1.$$

Hence show that neither the geometric distribution $1-F(x)=\exp(-\lfloor x\rfloor)$, for positive x, nor the Poisson distribution $1-F(x)=\sum_{k>x}\exp(-\lambda)\lambda^k/k!$, for positive x, are in the Gumbel domain of attraction.

Note: the condition above is true for all GEV distributions. Hence the geometric and Poisson distributions are not in the domain of attraction for any GEV.

Solution:

If $F\in\mathcal{D}(\Lambda)$, then there exists $z_0<x^*$, positive constant c, and absolutely continuous positive function f, with $f'\to 0$, such that for all $x>z_0$

$$1-F(x)=c\exp\left(-\int_{z_0}^{x}\frac{1}{f(t)}\,\mathrm{d}t\right),$$

for $z_0<x<x^*$. It is clear that the function $1-F$ is continuous for $x>z_0$. Hence, the limit of $1-F$ at some fixed t is the same from above and below, and so

$$\lim_{t\to x^*}\frac{\lim_{s\uparrow t}1-F(s)}{\lim_{s\downarrow t}1-F(s)}=\lim_{t\to x^*}\frac{1-F(t)}{1-F(t)}=\lim_{t\to x^*}1=1.$$

Now consider the survival function of the geometric distribution $1-F(s)=\exp(-\lfloor s\rfloor)$, for positive s. The function $1-F(s)$ is constant at every non-integer t. However, if we fix some positive integer n, then it is clear to see that

$$\lim_{n\to x^*}\frac{\lim_{s\uparrow n}1-F(s)}{\lim_{s\downarrow n}1-F(s)}=\lim_{n\to\infty}\frac{\exp(-n+1)}{\exp(-n)}=\mathrm{e}^{-1}\neq 1.$$

The geometric distribution is hence not in the Gumbel domain of attraction. Now consider the Poisson distribution which has survival function

$$1-F(x)=\mathrm{e}^{-\lambda}\sum_{k=\lfloor x\rfloor+1}^{\infty}\frac{\lambda^k}{k!}=1-\frac{\Gamma(\lfloor k+1\rfloor,\lambda)}{\lfloor k\rfloor!}.$$

Once again the distribution is continuous at non-integer valued x. Now take n to be a positive integer. Then

$$\lim_{s\uparrow n}1-F(s)=\mathrm{e}^{-\lambda}\sum_{k=n}^{\infty}\frac{\lambda^k}{k!}$$

$$\text{and }\lim_{s\downarrow n}1-F(s)=\mathrm{e}^{-\lambda}\sum_{k=n+1}^{\infty}\frac{\lambda^k}{k!}=\left[\lim_{s\uparrow n}1-F(s)\right]-\mathrm{e}^{-\lambda}\frac{\lambda^n}{n!}.$$

Observe that

$$\begin{aligned}\frac{\sum_{\ell=n}^{\infty}\lambda^{\ell}/\ell!}{\sum_{k=n+1}^{\infty}\lambda^{k}/k!} &= 1+\frac{\lambda^{n}/n!}{\sum_{k=n+1}^{\infty}\lambda^{k}/k!}\\ &= 1+\frac{\lambda^{n}}{n!}\frac{1}{\frac{\lambda^{n+1}}{(n+1)!}+\frac{\lambda^{n+2}}{(n+2)!}+\cdots}\\ &= 1+\frac{\lambda^{n}}{n!}\frac{1}{\frac{\exp(\lambda)}{n!}\gamma(n+1,\lambda)}\\ &= 1+\lambda^{n}\mathrm{e}^{-\lambda}\frac{1}{\gamma(n+1,\lambda)},\end{aligned}$$

where $\gamma(n+1,\lambda)=\int_0^{\lambda}t^{n+1-1}\exp(-t)\,\mathrm{d}t$ is the lower incomplete Gamma function. Now

$$\frac{\gamma(n+1,\lambda)}{\lambda^{n}\exp(-\lambda)}=\lambda^{-n}\mathrm{e}^{\lambda}\int_0^{\lambda}t^{n}\mathrm{e}^{-t}\,\mathrm{d}t\le\lambda^{-n}\mathrm{e}^{\lambda}\int_0^{\lambda}t^{n}\,\mathrm{d}t=\lambda^{-n}\mathrm{e}^{\lambda}\frac{\lambda^{n+1}}{(n+1)}\to 0.$$

Then

$$1+\lambda^{n}\mathrm{e}^{-\lambda}\frac{1}{\gamma(n+1,\lambda)}=1+\left[\frac{\gamma(n+1,\lambda)}{\lambda^{n}\exp(-\lambda)}\right]^{-1}\to+\infty.$$

It follows that the Poisson distribution is not in the maximal domain of attraction of the Gumbel distribution.