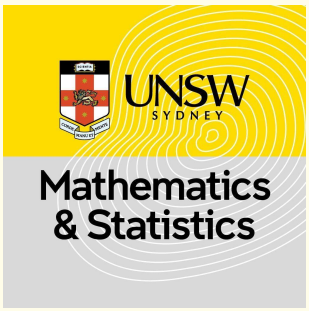


MATH5805
A Modern Introduction to Extreme Value Theory
Term 1, 2025

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Week 4 - Tutorial Solutions

Exercise 1

Let X be a Fréchet(α) distributed random variable, for some positive α. Let $Y = X - u$, for some sufficiently large u. Show that $Y|\{X > u\}$ follows a Generalised Pareto Distribution.

Hint: Note that

$$\frac{1 - \exp\left(-(u+y)^{-\alpha}\right)}{1 - \exp\left(-u^{-\alpha}\right)} \sim \left(1 + \frac{y}{u}\right)^{-\alpha}.$$

Solution:
Recall that X has distribution function

$$\Phi_\alpha(x) = \mathbb{P}\left(X \leq x\right) = \begin{cases} 0, & \text{for } x \leq 0 \\ \exp\left(-x^{-\alpha}\right), & \text{for } x > 0 \end{cases},$$

where α is positive, and that the GPD has survival function

$$1 - H_\xi(z) = \begin{cases} (1+\xi z)^{-1/\xi}, & \text{for } \xi \neq 0 \\ \exp(-z), & \text{for } \xi = 0 \end{cases},$$

with support respectively defined on $[0, \infty)$ and $[0, -\xi^{-1}]$ for $\xi \geq 0$ and $\xi < 0$. Recall that, for positive y, Y has conditional survival function

$$\begin{aligned} \mathbb{P}\left(Y > y | X > u\right) &= \mathbb{P}\left(X > u + y | X > u\right) \\ &= \frac{1 - F(u+y)}{1 - F(u)} \\ &= \frac{1 - \exp\left(-(u+y)^{-\alpha}\right)}{1 - \exp\left(-u^{-\alpha}\right)} \\ &\sim \left(1 + \frac{y}{u}\right)^{-\alpha}, \end{aligned}$$

which is equivalent to the survival function of the GPD with respective shape and scale parameters $\xi = \alpha^{-1}$ and $\sigma = u$.

Exercise 2

Let F be a smooth distribution function. Use Taylor's theorem, or otherwise, to show that $\log(F(x)) \approx -[1-F(x)]$ for large x.

Solution:
Observe that $F(z) = \exp(\log[F(z)])$. Applying the power series expansion of the exponential function yields

$$F(z) = \exp(\log[F(z)]) = 1 + \log[F(z)] + O\left(\log^2[F(z)]\right).$$

Let ε be a small positive quantity such that $F(z) = 1-\varepsilon$ for sufficiently large z. Then, noting that $\log(x) < x-1$ for all positive x, we have that

$$\log[F(z)] < F(z) - 1 = -\varepsilon.$$

It follows that $O\left(\log^2[F(z)]\right)$ will also be sufficiently small. The result is hence obtained upon rearranging the resulting approximation

$$F(z) \approx 1 + \log[F(z)].$$

Exercise 3

Let X be a GPD(μ, σ, ξ) distributed random variable whose distribution function is

$$F_\xi(x) = \begin{cases} 1 - \left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}, & \text{for } \xi \neq 0 \\ 1 - \exp\left(-\frac{x-\mu}{\sigma}\right), & \text{for } \xi = 0 \end{cases},$$

with respective support $z \geq 0$ and $0 \leq z \leq -1/\xi$ for non-negative and negative ξ, where $z = (x-\mu)/\sigma$. Show that

$$\mathbb{E}[X] = \begin{cases} \mu + \frac{\sigma}{1-\xi}, & \text{for } \xi < 1 \\ \infty, & \text{for } \xi \geq 1 \end{cases}.$$

Hint: We have that

$$\mathbb{E}[X] = \int_0^\infty 1 - F_X(s)\,\mathrm{d}s - \int_{-\infty}^0 F_X(t)\,\mathrm{d}t.$$

Solution:
Firstly, note that the GPD is a location-scale family. Hence, if $Y \sim \text{GPD}(0, 1, \xi)$, then $X \overset{d}{=} \mu + \sigma Y$. Also note that Y has support on $[0, \infty)$. Observe that Y is equivalent to the standard Exponential distribution when $\xi = 0$. Hence $\mathbb{E}[Y|\xi = 0] = 1$. Now let $\xi < 0$ and write $\alpha = -1/\xi > 0$. Supplying the substitution $u = 1 - s/\alpha$, we have that

$$\begin{aligned} \mathbb{E}[Y|\xi < 0] &= \int_0^\infty 1 - F_Y(s)\,\mathrm{d}s - \int_{-\infty}^0 F_Y(t)\,\mathrm{d}t \\ &= \int_0^\alpha \left(1 - \frac{s}{\alpha}\right)^\alpha \mathrm{d}s \\ &= \alpha \int_0^1 u^\alpha \,\mathrm{d}u \\ &= \frac{\alpha}{\alpha + 1} \\ &= \frac{1}{1-\xi}. \end{aligned}$$

Now take $\xi > 0$ and let $\alpha = \xi^{-1} > 0$. Utilising the substitution $v = 1 + s/\alpha$ then yields

$$\begin{aligned}
\mathbb{E}\left[Y | \xi > 0\right] &= \int_0^\infty 1 - F_Y(s)\,\mathrm{d}s - \int_{-\infty}^0 F_Y(t)\,\mathrm{d}t \\
&= \int_0^\infty \left(1 + \frac{s}{\alpha}\right)^{-\alpha} \mathrm{d}s \\
&= \alpha \int_1^\infty v^{-\alpha}\,\mathrm{d}v \\
&= \alpha \lim_{M\to\infty} \int_1^M v^{-\alpha}\,\mathrm{d}v \\
&= \frac{\alpha}{\alpha - 1}\left(\lim_{M\to\infty} 1 - M^{1-\alpha}\right),
\end{aligned}$$

which converges only if $\alpha \geq 1$. However, note that the expectation is infinite at $\alpha = 1$, taking the convention that $1/0 = +\infty$. Hence, we require that $\alpha > 1$, which implies that $\xi < 1$. The expectation then simplifies to $\alpha/(\alpha - 1) = 1/(1 - \xi)$. Finally, we conclude that

$$\mathbb{E}\left[X\right] = \mu + \sigma\mathbb{E}\left[Y\right] = \begin{cases} \mu + \frac{\sigma}{1-\xi}, & \text{for } \xi < 1 \\ +\infty, & \text{for } \xi \geq 1 \end{cases}.$$

Exercise 4

Verify the derivation of $\Delta z_m^\top$ given on slide 37 of the Week 4 Lecture material, for $\xi = 0$. Recall that the return level z_m can be written

$$z_m = \begin{cases} u + \tilde{\sigma}\xi^{-1}\left[(m\eta_u)^\xi - 1\right], & \text{for } \xi \neq 0 \\ u + \tilde{\sigma}\log\left(m\eta_u\right), & \text{for } \xi = 0 \end{cases}.$$