MATH5805
A Modern Introduction to Extreme Value Theory
Term 1, 2025

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Week 3 - Tutorial Solutions

Exercise 1

Let X be a Fréchet(α) distributed random variable, for some positive α, which has distribution function

$$\Phi_\alpha(x) = \begin{cases} 0, & \text{for } x \leq 0 \\ \exp\left(-x^{-\alpha}\right), & \text{for } x > 0 \end{cases}.$$

a) Show that X is in the domain of attraction for Φ, that is, $F \in \mathcal{D}(\Phi)$.

Solution:
It is sufficient to show that $1 - \Phi_\alpha$ is regularly varying. Through an application of L'Hôpital's rule, for some positive u, we have that

$$\begin{aligned} \lim_{x\to\infty} \frac{1-\Phi_\alpha(ux)}{1-\Phi_\alpha(x)} &= \lim_{x\to\infty} \frac{1-\exp\left(-(ux)^{-\alpha}\right)}{1-\exp\left(-x^{-\alpha}\right)} \\ &= \lim_{x\to\infty} \frac{-\alpha(ux)^{-\alpha}\exp\left(-(ux)^{-\alpha}\right)/x}{-\alpha(x)^{-\alpha}\exp\left(-x^{-\alpha}\right)/x} \\ &= u^{-\alpha} \lim_{x\to\infty} \exp\left(x^{-\alpha}\left(1-u^{-\alpha}\right)\right) \\ &= u^{-\alpha}, \end{aligned}$$

which implies that $1 - \Phi_\alpha$ is regularly varying with index $-\alpha$. Hence $\Phi_\alpha \in \mathcal{D}(\Phi)$ since $1 - \Phi_\alpha \in \text{RV}_{-\alpha}$.

b) Recall that the tail quantile function U for the distribution function F is defined as the left-continuous inverse

$$\begin{aligned} &U : [1,\infty) \to [x_*, x^*] \\ &U(y) = \left(\frac{1}{1-F}\right)^{-1}(y) = \inf\left\{x \in [x_*, x^*] : \frac{1}{1-F(x)} \geq y\right\}. \end{aligned}$$

Show that the left inverse for Φ_α is

$$U(y) = \left[\log\left(1 + \frac{1}{y-1}\right)\right]^{-1/\alpha}.$$

Solution:
Observe that $[x_*, x^*] = [0, +\infty]$ and that Φ_α is bijective in $[x_*, x^*]$. Hence, for $U(y) \in [x_*, x^*]$,

we may write that

$$F(U(y)) = 1 - \frac{1}{y}$$
$$\implies \exp\left(-U^{-\alpha}(y)\right) = 1 - \frac{1}{y}.$$

Rearranging with respect to U then yields the result.

c) Compute the norming sequences for Φ_α.

Solution:
Recall that the respective and location norming constants for distribution functions in $\mathcal{D}(\Phi)$ are $a_n = U(n)$ and $b_n = 0$. Hence, for $n > 1$,

$$a_n = U(n) = \left[\log\left(1 + \frac{1}{n-1}\right)\right]^{-1/\alpha}.$$

Exercise 2

Show that the Uniform(a, b) distribution, for constants a and b with $a < b$, is in the domain of attraction for Ψ and compute the norming sequences.

Solution:
Firstly, observe that $x^* = b < \infty$. It then suffices to show that $1 - F(b - 1/x)$ is regularly varying with index $-\alpha < 0$, where

$$F(x) = \begin{cases} 0, & \text{for } x \le a \\ \frac{x-a}{b-a}, & \text{for } a < x < b \\ 1, & \text{for } x \ge b \end{cases}.$$

For $x > (b-a)^{-1}$, we have that

$$\ell(x) := 1 - F\left(b - \frac{1}{x}\right) = 1 - \left[\frac{b - x^{-1} - a}{b-a}\right] = \frac{1}{x}\frac{1}{b-a}.$$

It follows that ℓ is a regularly varying function with index $\gamma = -1$, since, for some positive u,

$$\lim_{x\to\infty} \frac{\ell(ux)}{\ell(x)} = \lim_{x\to\infty} \frac{[ux(b-a)]^{-1}}{[x(b-a)]^{-1}} = u^{-1}.$$

Hence, $F \in \mathcal{D}(\Psi)$. Observe that $x_* = a$ and that F is bijective in (a, b). Hence, for $U(y) \in [a, b]$, we may write that

$$F(U(y)) = \frac{U(y) - a}{b-a} = 1 - \frac{1}{y} \implies U(y) = a + (b-a)\left(1 - \frac{1}{y}\right).$$

The respective scale and location norming sequences are hence

$$a_n = x^* - U(n) = b - \left[a + (b-a)\left(1 - \frac{1}{n}\right)\right] = \frac{b-a}{n}$$

and $b_n = b$.

Exercise 3

a) Suppose that $F \in \mathcal{D}(G_\xi)$, for some ξ, and that a_n and b_n are the respective scale and location norming constants. Explain why $F(a_n x + b_n) \to 1$ for all x which satisfies $0 < G_\xi(x) < 1$.

Solution:
Let M_n denote the maximum of a sequence of n independent and identically distributed random variables. Recall that $M_n \xrightarrow{\mathbb{P}} x^*$. Now $F(a_n x + b_n) \to 1$ if $a_n x + b_n \to x^*$. Suppose that $a_n x + b_n$ does *not* converge to x^*, where we specify that if $a_n x + b_n$ is a convergent sequence, then its limit $L < x^*$. This implies that

$$\mathbb{P}(M_n \leq a_n x + b_n) = F^n(a_n x + b_n) \to 0.$$

However, the limit for $F^n(a_n x + b_n)$ is not degenerate, by assumption, yielding a contradiction. Hence, we have that $a_n x + b_n \to x^*$, which implies that $F^n(a_n x + b_n) \to 1$.

b) Let X and $Y = -1/X$ be random variable with respective distribution functions F_X and F_Y. Show that if F_X is in the domain of attraction for Φ, then F_Y is in the domain of attraction for Ψ, that is, $F_X \in \mathcal{D}(\Phi) \implies F_Y \in \mathcal{D}(\Psi)$.

Hint: The previous question implies that $-\log[F(a_n x + b_n)] \sim 1 - F(a_n x + b_n)$.

Solution:
We can express F_Y in terms of F_X. and *vice versa*. We have that

$$1 - F_Y(y) = \mathbb{P}(Y > y) = \mathbb{P}\left(-X^{-1} > y\right) = \mathbb{P}\left(X > -y^{-1}\right) = 1 - F_X\left(-\frac{1}{y}\right).$$

From our assumption, there exists positive and real sequences a_n and b_n such that $F_X^n(a_n x + b_n) \to \Phi_\alpha(x)$. It follows that

$$n \log[F_X(a_n x + b_n)] \sim \log[\Phi_\alpha(x)] = -x^{-\alpha}.$$

Utilising corollaries from the previous question (i.e. using the hint), we equivalently have that

$$n[1 - F_X(a_n x + b_n)] = n\left[1 - F_Y\left(-\frac{1}{a_n x + b_n}\right)\right] \sim -x^{-\alpha}.$$

Suppose that there respectively exists positive and real sequences c_n and d_n such that $F_Y \in \mathcal{D}(G_\xi)$. It follows that

$$n[1 - F_Y(c_n y + d_n)] \sim -\left(-\frac{1}{y}\right)^{-\alpha} = -(-y)^\alpha,$$

for positive α. Supplying the log equivalence and taking exponentials then shows that, in fact, $G_\xi \equiv \Psi_\alpha$, satisfying the result.

Exercise 4

Show that the set of positive polynomials of order n are regularly varying and compute its index.

Solution:
Let $(a_k)_{k=0}^n$ be a sequence of constants such that for all positive x, $p_n(x) = \sum_{k=0}^n a_k x^k > 0$. Then,

factorising and simplifying for x^n and u^n, for some positive u, we have that

$$\begin{aligned}\lim_{x\to\infty}\frac{p_n(ux)}{p_n(x)} &= \lim_{x\to\infty}\frac{a_n(ux)^n + a_{n-1}(ux)^{n-1} + \cdots + a_0}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}\\ &= u^n \lim_{x\to\infty}\frac{a_n + a_{n-1}\,(ux)^{-1} + \cdots + a_0\,(ux)^{-n}}{a_n + a_{n-1}x^{-1} + \cdots + a_0 x^{-n}}\\ &= u^n.\end{aligned}$$