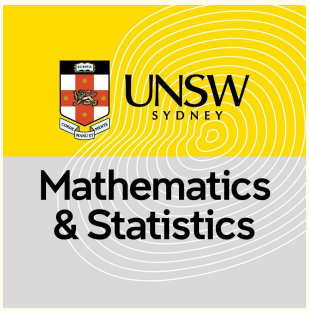


MATH5805
A Modern Introduction to Extreme Value Theory
Term 1, 2025

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Week 2 - Tutorial Solutions

Exercise 1

Show that the standard Gumbel, Frechet, and Weibull distributions are max-stable and determine their norming constants.

Solution:
We must show that the maximum of a sequence of random variables is closed in distribution for any finite length of the sequence. This contrasts with showing that a distribution is in some domain of attraction, which is a limit result. Let us first consider the standard Gumbel random variable X, which has distribution function

$$\Lambda(x) = \mathbb{P}(X \leq x) = \exp(-\exp(-x)).$$

Let $X_1, \dots, X_n$ be an independent sequence of random variables, each of which has distribution function Λ. Let $M_n = \max\{X_1, \dots, X_n\}$, and let a_n and b_n be positive and real constants. Noting that $\exp(-\log n) = n^{-1}$, we observe that Λ is max-stable with respect to the sequences $a_n = 1$ and $b_n = \log n$, since

$$\begin{aligned}
\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) &= \Lambda^n(a_n x + b_n) \\
&= \exp^n(-\exp[-(a_n x + b_n)]) \\
&= \exp(-n\exp[-(x + \log n)]) \\
&= \exp(-n\exp(-x)\exp(-\log n)) \\
&= \Lambda(x).
\end{aligned}$$

Now consider the standard Fréchet random variable Y, which has distribution function

$$\Phi(y) = \mathbb{P}(Y \leq y) = \begin{cases} 0, & \text{for } y \leq 0 \\ \exp\left(-\frac{1}{y}\right), & \text{for } y > 0 \end{cases}.$$

Let $Y_1, \dots, Y_n$ be an independent sequence of random variables, each of which has distribution function Φ. Let $N_n = \max\{Y_1, \dots, Y_n\}$, and let c_n and d_n be positive and real constants. It follows

that Φ is max-stable with respect to the sequences $c_n = n$ and $d_n = 0$ since, for positive y,

$$\begin{aligned}\mathbb{P}\left(\frac{N_n - d_n}{c_n} \leq y\right) &= \Phi^n(c_n y + d_n) \\ &= \exp^n\left(-\frac{1}{c_n y + d_n}\right) \\ &= \exp\left(-\frac{n}{ny + 0}\right) \\ &= \Phi(y).\end{aligned}$$

Finally, let Z be a standard Weibull random variable, which has distribution function

$$\Psi(z) = \mathbb{P}(Z \leq z) = \begin{cases}\exp(z), & \text{for } z \leq 0 \\ 1, & \text{for } z > 0\end{cases}.$$

Let $Z_1, \ldots, Z_n$ be an independent sequence of random variables, each of which has distribution function Ψ. Let $O_n = \max\{Z_1, \ldots, Z_n\}$, and let e_n and f_n be positive and real constants. Then Ψ is max-stable with respect to the sequences $e_n = n^{-1}$ and $f_n = 0$ since, for negative z,

$$\begin{aligned}\mathbb{P}\left(\frac{O_n - e_n}{f_n} \leq z\right) &= \Psi^n(e_n z + f_n) \\ &= \exp^n(e_n z + f_n) \\ &= \exp\left(n\left[\frac{1}{n}z + 0\right]\right) \\ &= \Psi(z).\end{aligned}$$

Exercise 2

Show that $\mathcal{D}(\Lambda) \ni F(x) = 1 - \exp(-x^\alpha)$, for positive x and α. Compute the norming constants.

Solution:
Recall that $F \in \mathcal{D}(\Lambda)$ if, and only if, F is *von Mises*, in which there exists $z_0 < x^* = +\infty$, a positive constant c, and positive function f, with $f' \to 0$, such that

$$1 - F(x) = c\exp\left(-\int_{z_0}^{x} \frac{1}{f(u)}\,\mathrm{d}u\right).$$

Let $z_0 = 0$, $c = 1$, and $f(u) = \alpha^{-1}u^{1-\alpha}$. We have chosen f such that $1/f$ is the derivative for u^α. We also have that $f' \to 0$. Then

$$c\exp\left(-\int_{z_0}^{x} \frac{1}{f(u)}\,\mathrm{d}u\right) = \exp\left(-\int_0^x \alpha u^{\alpha-1}\,\mathrm{d}u\right) = \exp\left(-x^\alpha\right),$$

which satisfies exactly for all non-negative z_0. We chose f here through direct means. In the lecture it was shown that, at least in the tail, you can choose

$$f = \frac{1 - F}{F'}.$$

Check to see that f is in fact the same here. Recall that the respective scale and location norming constants can be written

$$a_n = g(U(n)) \text{ and } b_n = U(n),$$

where $g = (1-F)/F'$ and $U = (1/(1-F))^{-1}$ taken to be the left continuous inverse. Noting that F is one-to-one on its support, we have that

$$F(U(y)) = 1 - \exp\left(-U^{\alpha}(y)\right) = 1 - \frac{1}{y} \implies U(y) = \log^{1/\alpha}(y).$$

We also have that

$$g(x) = \frac{1-F(x)}{F'(x)} = \frac{\exp\left(-x^{\alpha}\right)}{\alpha x^{\alpha-1}\exp\left(-x^{\alpha}\right)} = \frac{1}{\alpha}x^{1-\alpha}.$$

Hence,

$$a_n = \frac{1}{\alpha}\log^{-(1-1/\alpha)}(n) \text{ and } b_n = \log^{1/\alpha}(n)$$

satisfy as norming constant.

Exercise 3

Determine if the following functions are regularly varying and compute their index γ where applicable.

1. $\ell(x) = \mathrm{e}^{kx}$, for some positive k.

   Solution:
   For some positive u, we have that

   $$\lim_{x\to\infty}\frac{\ell(ux)}{\ell(x)} = \lim_{x\to\infty}\frac{\exp(kxu)}{\exp(kx)} = \begin{cases} 0, & \text{for } u \le 1 \\ +\infty, & \text{for } u > 1 \end{cases}.$$

   Hence, $\ell(x) = \mathrm{e}^{kx}$ is not regularly varying since the condition must be satisfied for all positive u.

2. $\ell(x) = \sin(x) + 2$.

   Solution:
   It is clear to see that ℓ is not regularly varying since the asymptotic limit for $\sin(x)$ is indeterminate.

3. $\ell(x) = \exp\left(\log^{k}(x)\right)$.

   Hint: We have that

   $$\lim_{y\to 0}\frac{(1+y)^k - 1}{y} = k,$$

   and that $f \sim g \equiv f(y) = g(y)(1+o(1))$.

   Solution:

Writing $y = \log(u)/\log(x)$, for some positive u, we have that

$$\begin{aligned}
\frac{\ell(ux)}{\ell(x)} &= \frac{\exp\left(\log^k(ux)\right)}{\exp\left(\log^k(x)\right)} \\
&= \exp\left(\log^k(ux) - \log^k(x)\right) \\
&= \exp\left([\log(u) + \log(x)]^k - \log^k(x)\right) \\
&= \exp\left(\log^k(x)\left[\frac{\log(u)}{\log(x)} + 1\right]^k - \log^k(x)\right) \\
&= \exp\left(\log^k(x)\left[(y+1)^k - 1\right]\right).
\end{aligned}$$

Then, for an decreasing positive function f, we have that

$$\lim_{x\to\infty} \frac{(1+f(x))^k - 1}{f(x)} = \lim_{y\to 0} \frac{(1+y)^k - 1}{y}.$$

Hence, we have that

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ell(ux)}{\ell(x)} &= \exp\left(\lim_{x\to\infty} \log^k(x)\, ky(1+o(1))\right) \\
&= \exp\left(ky \lim_{x\to\infty} \log^k(x)\right) \\
&= \exp\left(k\log(u) \lim_{x\to\infty} \log^{k-1}(x)\right).
\end{aligned}$$

Now

$$\lim_{x\to\infty} \log^{k-1}(x) = \begin{cases} 0, & \text{for } k < 1 \\ 1, & \text{for } k = 1 \\ +\infty, & \text{for } k > 1 \end{cases}.$$

It follows that ℓ is *not* regularly varying for $k > 1$. However, ℓ is slowly varying for $k < 1$, and regularly varying with index $\gamma = 1$ for $k = 1$.