MATH5805 – A Modern Introduction to Extreme Value Theory

Week 2 - Univariate Extremes: the Block Maxima approach (I)

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Lecture outline

1. Introduction
2. Limit Laws
3. Proof of the Extremal Type Theorem
4. Using the Extremal Type Theorem

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1. Introduction



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Goals



Week 2 & 3 are about introducing the **Block Maxima** approach.

This week, we

- Familiarise ourselves with some important concepts of (univariate) Extreme Value Theory such as **max-stability** and **tail index**
- Introduce the **Generalised Extreme Value distribution**

Note that throughout this lecture we will make the assumption that $X_1, \ldots, X_n$ is a sequence of i.i.d. random variables with **common distribution function** F.

Block Maxima - Overview

As its name indicates, we consider the **maximum of a "block"** of say, size n:

$$M_n = \max_{i=1,\dots,n} X_i$$

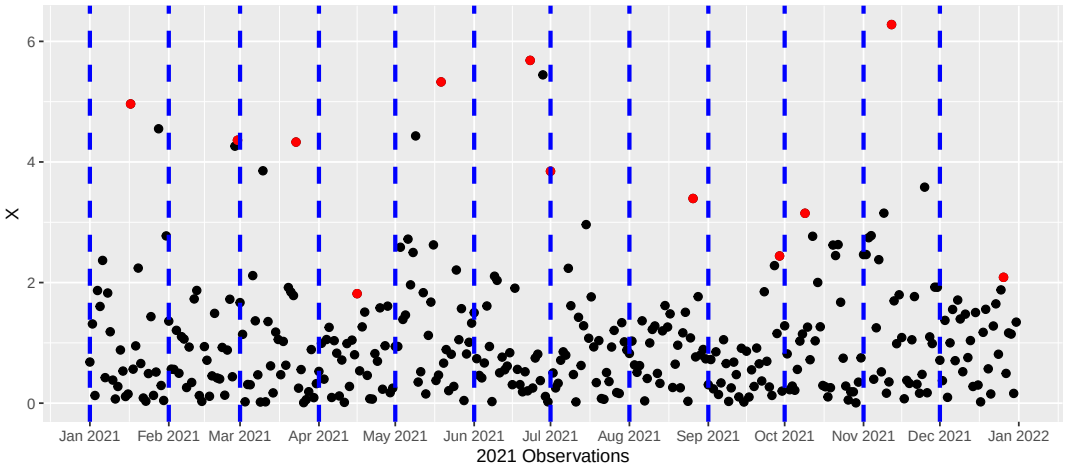
Assuming a dataset of size $n \times m$, it is divided into **m blocks** of **size n**. We obtain a sequence of (only) m maxima that we can define as

$$\left(M_n^1, \dots, M_n^m\right), \quad M_n^j = \max_{i \in \text{ bloc } j} X_i$$

Example

Consider the daily concentration level of an air pollutant in 2021 ($n \times m = 365$). Focus on the monthly maximum concentration (**m=12**). See next slide.

Block Maxima - Overview



Simulated daily levels of an air pollutant in 2021. Red dots represent monthly maxima.

Block Maxima - Overview



Assuming n is finite, what is the distribution of M_n? (1st year exercise!)

$$\begin{aligned} F_{M_n}(x) = \mathbb{P}(M_n \leq x) &= \mathbb{P}(\max\{X_1, \ldots, X_n\} \leq x) \\ &= \mathbb{P}(X_1 \leq x, \ldots, X_n \leq x) \\ &= \mathbb{P}(X_1 \leq x) \times \cdots \times \mathbb{P}(X_n \leq x) \\ &= \mathbb{P}(X \leq x)^n = F^n(x). \end{aligned} \tag{1}$$

- So if we know F, we can model M_n.
- **But what if F is unknown?**
- For finite n we are stuck.
- But CLT works for unknown F as $n \to \infty$
- **Can we do something similar for M_n as $n \to \infty$?** (Answer = yes!)



2. Limit Laws

There **one obvious issue** with (1) when $n \to \infty$: if $x_0 = \sup\{x : F(x) < 1\}$ then

$$\mathbb{P}(M_n \leq x_0) = F^n(x_0) \to 0, \quad \textbf{as } n \to \infty$$

$\implies$ **distribution of M_n degenerates to point mass at x_0**

There is a need for **rescaling** and thus we consider

$$M_n^* = \frac{M_n - b_n}{a_n},$$

where $a_n > 0$ and $b_n \in \mathbb{R}$ are appropriate norming constants.

Limit Laws

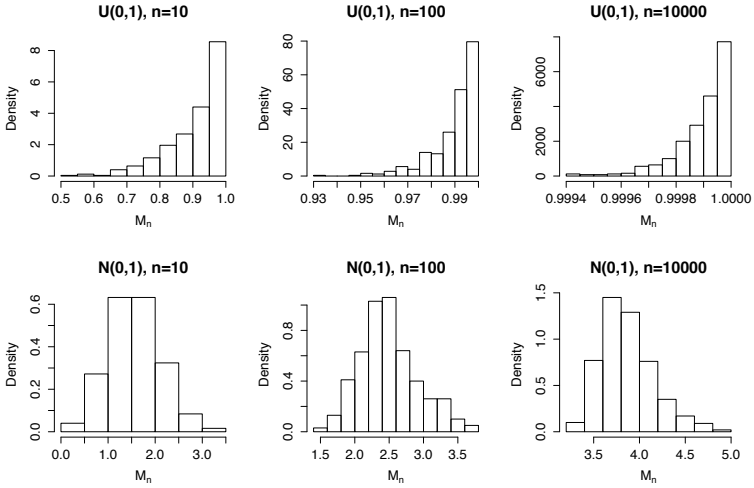


Figure 1: Density histogram of M_n when $n = 10, 100, 10000$ when $X \sim \mathcal{U}[0,1]$ (top) and $X \sim \mathcal{N}(0,1)$ (bottom)

The most famous weak convergence theorem

[CLT] Consider that X has mean μ and variance σ^2, then

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{w} \mathcal{N}(0,1)$$

Definition[Reminder]
Let $X_i, X, n \geq 1$ be real-valued random variables with distribution functions F_n and F. The sequence (X_n) **converges in distribution (= weakly)** to X if and only if for all continuity point x of F

$$F_n(x) \longrightarrow F(x), \quad \text{as } n \to \infty.$$

The most famous weak convergence theorem

The **CLT** provides a statement of the form:

$$\frac{h(X_1, \ldots, X_n) - b_n}{a_n} \xrightarrow{w} Y,$$

where $h(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i$, $b_n = n\mu$, $a_n = \sigma\sqrt{n}$ and $Y \sim \mathcal{N}(0, 1)$.



Is there an analogous result for $h(X_1, \ldots, X_n) = \max(X_1, \ldots, X_n)$?

The most famous weak convergence theorem

Example

Consider 2000 samples of size 50 from the Exponential distribution with unit rate, i.e. $X_1, \ldots, X_{50} \sim \mathrm{Exp}(1)$. As such, we have $\bar{X} = \frac{1}{\sqrt{50}} \left(\sum_{i=1}^{50} X_i - 50 \right) \xrightarrow{w} \mathcal{N}(0, 1)$

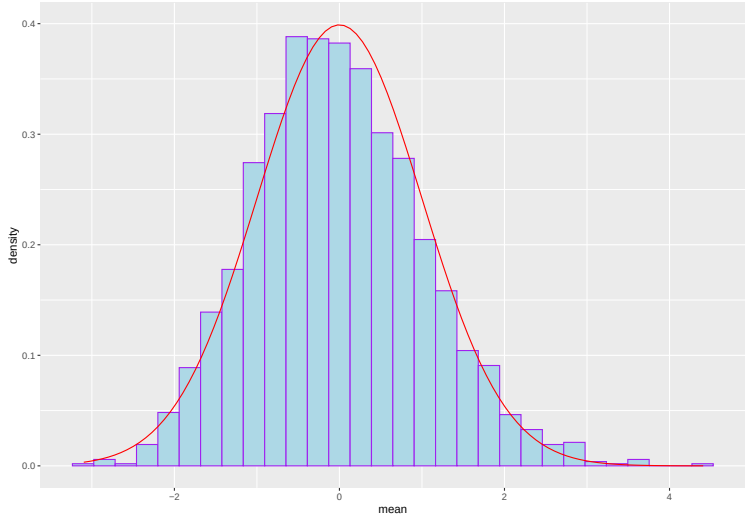
We illustrate this example in R as follows:

```
set.seed(321)
n <- 50
n.mean <- 2000
data <- matrix(rexp(n*n.mean),ncol=n.mean)
data.mean <- data.frame(mean=(apply(data,2,sum)-n)/sqrt(n))

ggplot(data.mean, aes(x=mean)) +
  geom_histogram(aes(y=after_stat(density)), colour="purple", fill="lightblue") +
  stat_function(fun = dnorm, args = list(mean = 0, sd = 1), colour="red")
```

The most famous weak convergence theorem

Which produces:



Extreme Value Theorem: limit law of maxima

Assume that there exists sequences of constants $(a_n) > 0$, $(b_n) \in \mathbb{R}$, and a non-degenerate random variable Y with d.f.$\sim G$, such that

$$M_n^* \xrightarrow{d} Y.$$

The d.f.'s G that can occur in the above limit are called **Extreme Value distributions**.

Theorem[Fisher & Tipett, 1928; Gnedenko 1943]
The only possible limiting distribution function G, up to location-scale transformation, is the **Generalized Extreme Value (GEV)** distribution, i.e.

$$\mathbb{P}(Y \leq y) := G_\xi(y) = \begin{cases} \exp\left\{-\left[(1+\xi y)_+\right]^{-1/\xi}\right\} & \xi \neq 0 \\ \exp\left\{-\exp\{-y\}\right\} & \xi = 0 \end{cases}, \quad \forall y \in \mathbb{R}$$

where $z_+ = \max(0, z)$.

Extreme Value Theorem: limit law of maxima

When $\xi = 0$, $G_0(y)$ corresponds in fact to $\lim_{\xi \to 0} G_\xi(y)$. **Assume this throughout.**

The GEV distribution is characterised by its shape parameter $\xi \in \mathbb{R}$, also known as the **extrema value index**, unifying three limiting distributions.

- **[Type I] Gumbel distribution**:

$$\Lambda(y) = G_0(y), y \in \mathbb{R}.$$

- **[Type II] Fréchet distribution**:

$$\Phi_\alpha(y) = \begin{cases} 0 & y \leq 0, \\ \exp\{-y^{-\alpha}\} & y > 0. \end{cases}$$

It corresponds to $G_\xi((y-1)/\xi)$ with $\xi = 1/\alpha$.

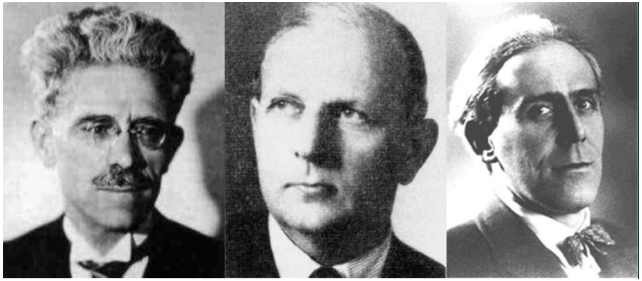
Extreme Value Theorem: limit law of maxima

- **[Type III] Weibull distribution**:

$$\Psi(y) = \begin{cases} \exp\{-(-y)^{\alpha}\} & y \leq 0, \\ 1 & y > 0. \end{cases}$$

It corresponds to $G_{\xi}(-(y+1)/\xi)$ with $\xi = -1/\alpha < 0$.

Shapes and sign of ξ: Fréchet, Weibull & Gumbel



- **Maurice Fréchet (1878–1973)**: French mathematician (topology, functional analysis, proba, statis). Identified the limit law of heavy-tailed distributions.
- **Walloddi Weibull (1887–1979)**: Swedish engineer & mathematician. Material fatigue.
- **Emil Julius Gumbel (1891–1966)**: German mathematician & political essayist. Left Germany after his expelling from Heidelberg (32) and rejoined France then the U.S.A. (40). Taught in Paris, Lyon, Columbia. Considered the as the 'father' of EVT.

GEV distribution

With a location $\mu \in \mathbb{R}$ and a scale $\sigma > 0$ parameter completing the parameter vector $\theta = (\mu, \sigma, \xi)$, the d.f. of the **GEV** becomes:

$$G(y|\theta) = \begin{cases} \exp\left\{-\left[1+\xi\left(\frac{y-\mu}{\sigma}\right)\right]_+^{-1/\xi}\right\} & \xi \neq 0 \\ \exp\left\{-\exp\{-\frac{y-\mu}{\sigma}\}\right\} & \xi = 0 \end{cases}, \quad \forall y \in \mathbb{R}$$

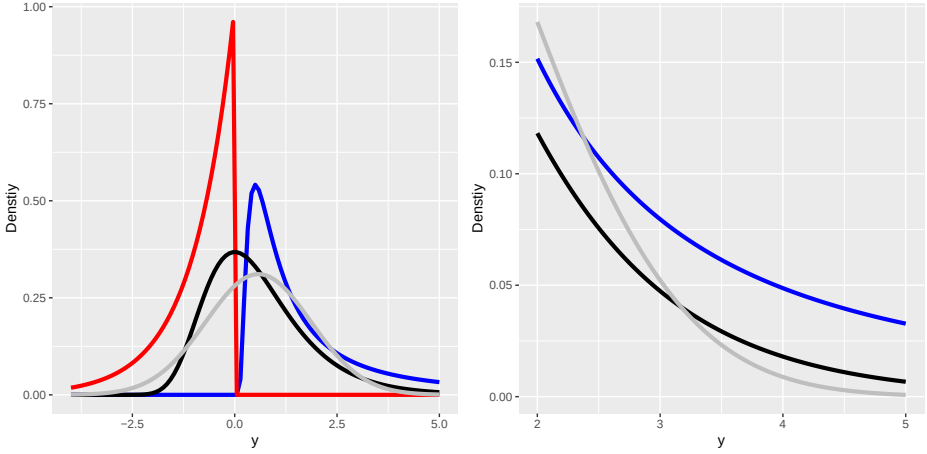
The corresponding **probability density function** (pdf) is:

$$g(y|\theta) = \frac{1}{\sigma} t^{\xi+1}(y|\theta) \exp\left\{-t(y|\theta)\right\},$$

where

$$t(y|\theta) = \begin{cases} \left[1+\xi\left(\frac{y-\mu}{\sigma}\right)\right]_+^{-1/\xi} & \xi \neq 0, \\ \exp\left\{-\frac{y-\mu}{\sigma}\right\} & \xi = 0. \end{cases}$$

GEV distribution



GEV densities with $\theta = (-1, 1, -1)$, $\theta = (1, 1, 1)$, $\theta = (0, 1, 0)$ and Gaussian density with same mean and variance as the Gumbel one. Right panel is a zoom on the tail.

GEV distribution



Exercise

Consider the three sets of parameters from the previous slide. Use `R` to generate series of 120 monthly maxima of an air pollutant from January 2011 to December 2021. Plot them and draw the bounds if needed.

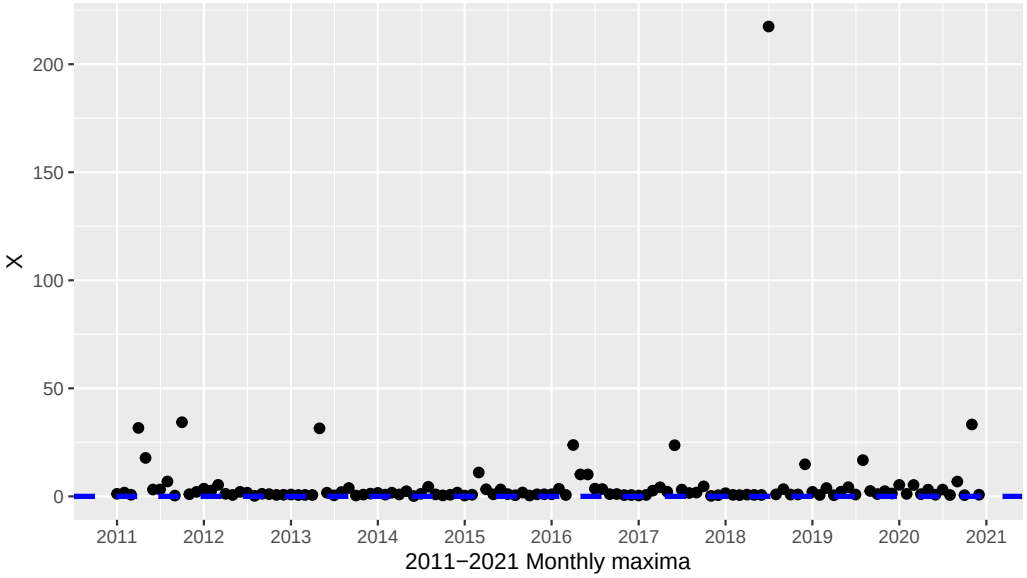
Hint: you might want to use the `rgev()` function from the `evd` `R` package.

Exercise - (partial) solution

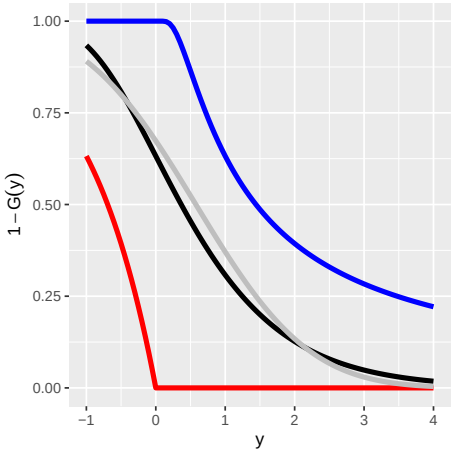
```
set.seed(123)
datesM = seq.Date(as.Date("2011/1/1"), by = "month", length.out = 120)
dat.fre <- data.frame(dat = rgev(120, loc = 1, scale = 1, shape=1),
                      month = datesM)

ggplot(dat.fre, aes(month,dat)) +
  geom_point(size=2) +
  scale_x_date(date_breaks = "1 year", date_labels =  "%Y") +
  xlab("2011-2021 Monthly maxima") +
  ylab("X") +
  geom_hline(yintercept=0, colour="blue", linetype="dashed", linewidth=1.2)
```

GEV distribution



GEV distribution



GEV survival function with $\theta = (-1, 1, -1)$, $\theta = (1, 1, 1)$, $\theta = (0, 1, 0)$ and Gaussian density with same mean and variance as the Gumbel one. Right panel is a zoom on the tail.

GEV distribution



Exercise

Assume that $X \sim \mathrm{GEV}(\mu, \sigma, \xi)$. We want to establish some properties of X.

1. Calculate $\mathrm{E}[X]$.
   Hint: Consider the case $\xi = 0$ and $\xi \neq 0$ with $\xi < 1$ and $\xi \geq 1$.
2. Calculate $\mathrm{Var}[X]$.
   Hint: Consider the case $\xi = 0$ and $\xi \neq 0$ with $\xi < 1/2$ and $\xi \geq 1/2$.



3. Proof of the Extremal Type Theorem

Max-stability

Definition

Consider $X_1, \ldots, X_n$ i.i.d. copies of a random variable X with $n \geq 2$, then X is said to be **stable** or to have a **stable distribution** if there exists some norming constant $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$X_1 + \cdots + X_n \stackrel{d}{=} a_n X + b_n,$$

where $\stackrel{d}{=}$ denotes equality in distribution.

Central Limit Theorem: the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is **asymptotically** normally distributed no matter the underlying distribution of X.

Max-stability

Example

It is well know that if $X \sim \mathcal{N}(\mu, \sigma^2)$ then $\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$.

It is thus clear that the Normal distribution is also **"mean-stable"**.



Can you think of another common distribution that is stable?

Max-stability

Following the analogy with the Central Limit Theorem, rather than being interested in the sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, we want to focus on the **sample maximum** $M_n = \max_{i=1,\dots,n}\{X_i\}$.

Definition Consider $X_1, \dots, X_n$ i.i.d. copies of a random variable X with $n \geq 2$, then X is said to be **max-stable** or to have a **max-stable distribution** if there exists some norming constant $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\max\{X_1, \cdots, X_n\} \stackrel{d}{=} a_n X + b_n.$$

Max-stability

Summary: the maximum of samples drawn from a max-stable distribution come from the same distribution, but with changed location and scale parameters.

Theorem

A distribution is max-stable if and only if it is a GEV distribution.

- Easy to check that Gumbel, Fréchet, Weibull are all max-stable.
- Converse is harder (but true).

Max-stability



Exercise

Consider the three sets of GEV parameters $\theta = (-1, 1, -1)$, $\theta = (1, 1, 1)$ and $\theta = (0, 1, 0)$, respectively corresponding to the Weibull, fréchet and Gumbel distributions.

Show analytically that these distributions are max-stable and derive the expression for the constants a_n and b_n.

Proof of the Extremal Type Theorem

Suppose that for large n

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq z\right) \approx G(z).$$

Hence for any integer $k \in \{1, 2, \ldots\}$, since nk is large

$$\mathbb{P}\left(\frac{M_{nk} - b_{nk}}{a_{nk}} \leq z\right) \approx G(z). \tag{2}$$

M_{nk} is max of k variables, each having the same distribution as M_n.

$$\mathbb{P}\left(\frac{M_{nk} - b_n}{a_n} \leq z\right) = \left[\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq z\right)\right]^k. \tag{3}$$

Proof of the Extremal Type Theorem

From (2) and (3) respectively we have

$$\mathbb{P}(\mathbf{M_{nk}} \leq \mathbf{z}) \approx \mathbf{G}\left(\frac{\mathbf{z} - \mathbf{b_{nk}}}{\mathbf{a_{nk}}}\right) \text{ and } \mathbb{P}(\mathbf{M_{nk}} \leq \mathbf{z}) \approx \mathbf{G^k}\left(\frac{\mathbf{z} - \mathbf{b_n}}{\mathbf{a_n}}\right).$$

G and G^k are identical, apart from location/scale parameters. **Hence G is max-stable, and therefore GEV.**

4. Using the Extremal Type Theorem



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4.1. Examples

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Example - Exponential(1)

The distribution function is $F(x) = 1 - e^{-x}$ for $x > 0$.

Let $a_n = 1$ and $b_n = \log n$. Then

$$\begin{aligned} \mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq z\right) &= F^n(z + \log n) \\ &= (1 - e^{-(z+\log n)})^n \\ &= (1 - e^{-z}/n)^n \\ &\to \exp(-e^{-z}) \qquad \text{as } n \to \infty. \end{aligned}$$

which is **Gumbel, with $\xi = 0$ in the GEV family.**

Extremal Type Theorem - Examples

Example - Exponential(1)

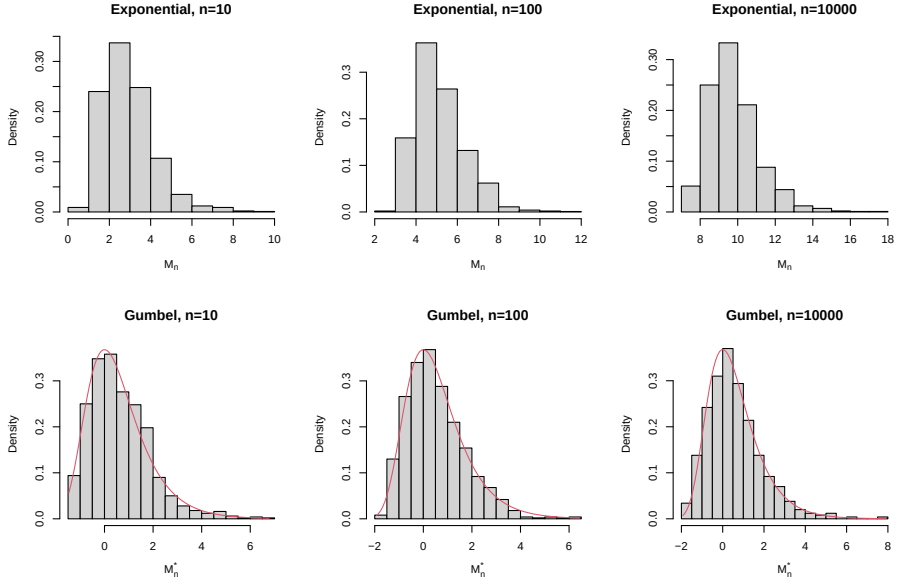
We provide an illustration in R:

```
set.seed(123)
n.seq <- c(10, 100, 10000)
rep <- 1000
mu <- matrix(NA, ncol=rep, nrow=length(n.seq))
for(i in 1:length(n.seq)) {
  u <- matrix(rexp(rep*n.seq[i]), ncol=rep)
  mu[i,] <- apply(u, 2, max)
}
```

Extremal Type Theorem - Examples

```
par(mfrow=c(2,3))
for(i in 1:3) hist(mu[i,], probability=T, xlab=expression(M[n]),
                   main=paste("Exponential, n=",get("n.seq")[i],sep=""))
for(i in 1:3) {
  a <- 1
  b <- log(n.seq[i])
  yy <- (mu[i,]-b)/a
  xx <- seq(min(yy),max(yy),length=100)
  yyy <- dgumbel(xx)
  hist(yy, probability=T, xlab=expression(M[n]^"*"),
       main=paste("Gumbel, n=",get("n.seq")[i],sep=""),
       ylim=c(0,max(yyy)), nclass=20)
  lines(xx, yyy, col=2, lwd=2)
}
```

Extremal Type Theorem - Examples



Example - Fréchet(1)

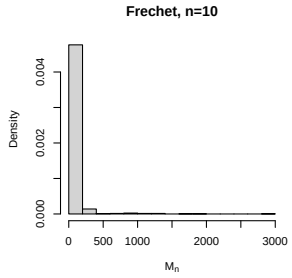
The distribution function is $F(x) = \exp(-1/x)$ for $x > 0$.

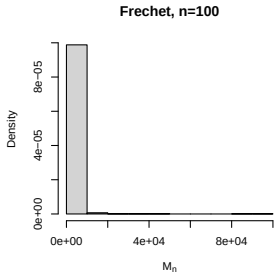
Let $a_n = n$ and $b_n = 0$. Then

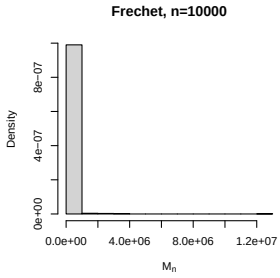
$$\begin{aligned} \mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq z\right) &= F^n(nz) \\ &= [\exp(-1/(nz))]^n \\ &= \exp(-1/z) \qquad \text{for any } n. \end{aligned}$$

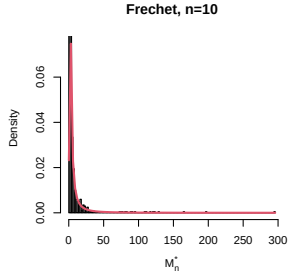
which is **standard Fréchet ($\xi = 1$ in the GEV family).**

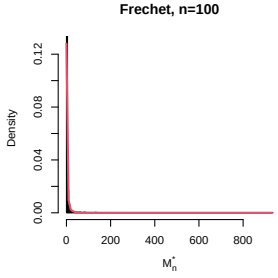
Extremal Type Theorem - Examples

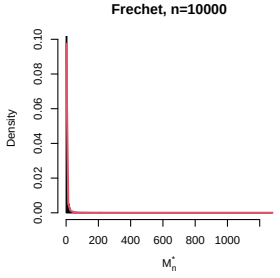












Extremal Type Theorem - Examples

Example - Uniform(0,1)

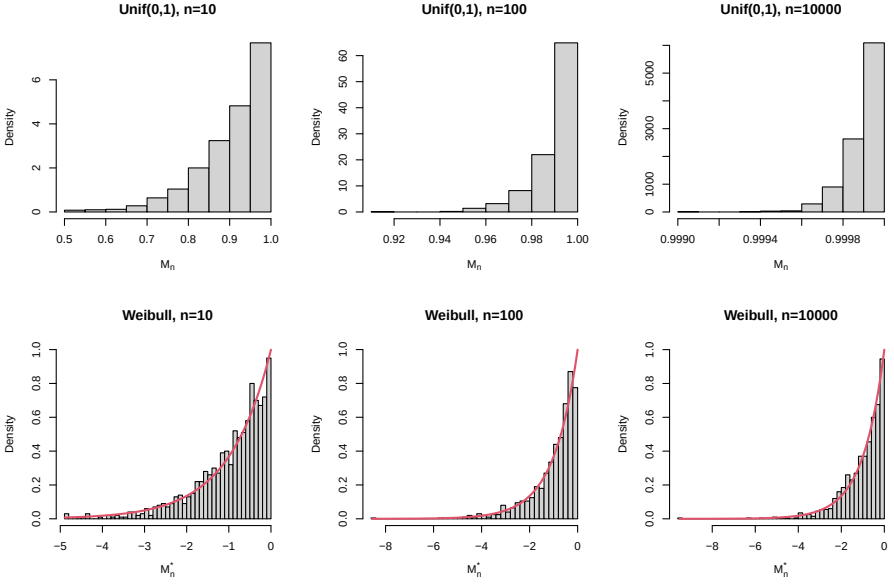
The distribution function is $F(x) = x$ for $0 \leq x \leq 1$.

Let $a_n = 1/n$ and $b_n = 1$. Hence $(M_n - b_n)/a_n \leq 0$. Then

$$\begin{aligned} \mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq z\right) &= F^n(z/n + 1) \qquad (\text{for } z < 0) \\ &= \left(1 + \frac{z}{n}\right)^n \\ &\to e^z \qquad \text{as } n \to \infty. \end{aligned}$$

which is **Weibull (finite upper end point), with $\xi = -1$ in the GEV family.**

Extremal Type Theorem - Examples



4.2. Domain of attraction

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Domain of attraction

Definition

Under the Extremal Type Theorem, i.e. whenever there exists sequences of constants $(a_n) > 0$ and $(b_n) \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F^n(a_n y + b_n) = G(y; \mu, \sigma, \xi),$$

F is said to be **in the (maximum) domain of attraction (MDA) of** G, denoted by

$$F \in \mathcal{D}(G).$$

Domain of attraction

Examples

- Exponential(1) is in the MDA of Gumbel $\Longrightarrow F \in \mathcal{D}(\Lambda)$.
- Fréchet(1) is in the MDA of Fréchet $\Longrightarrow F \in \mathcal{D}(\Phi)$.
- Uniform(0,1) is in the MDA of Weibull $\Longrightarrow F \in \mathcal{D}(\Psi)$.

In practice:

- **Gumbel domain:** light tail. Fits data from hydrology such as precipitation or annual maximum of water levels (dikes).
- **Fréchet domain:** heavy tail. Fits data from precipitation, stream flow and economic impacts (financial log-returns, Dow Jones etc.).
- **Weibull domain:** fInite right endpoint. Used to model extremes with an upper limit such as temperature, wind speed, sea level, athletic records and lifetimes.



Exercise

Show that if a random variable X has a distribution in the MDA of Fréchet, i.e. such that $F_X \in \mathcal{D}(\Phi)$ then $Y = -X^{-1}$ has a distribution in the MDA of the Weibull, i.e. $F_Y \in \mathcal{D}(\Psi)$.

Domain of attraction

Comments:

- The choice of a_n and b_n is not unique.
- Different choices (that lead to non-degenerate limits) are always in the GEV family with the same ξ parameter (it is just a location/scale shift).
- But with different location and scale parameters.

New Questions:

- How do we know the domain of attraction for a given F?
- How can we find a_n and b_n?

Domain of attraction

Let the upper and lower bounds of the support of a law F to be:

$$x^* = \sup_{x \in \mathbb{R}} \{x : F(x) < 1\} \quad \text{and} \quad x_* = \inf_{x \in \mathbb{R}} \{x : F(x) > 0\}.$$

It is of interest to

1. **check which domain of attraction G_ξ F is in**, i.e. $F \in \mathcal{D}(G_\xi)$
2. **find the norming constants $a_n > 0$ and $b_n \in \mathbb{R}$.**

We need conditions on the tail of F.

Domain of attraction

Theorem[dHF06, Theorem 1.2.5]

A unified sufficient and necessary condition for $F \in \mathcal{D}(G_\xi)$ is, for all x with $1+\xi x > 0$, there exists some positive function f such that

$$\lim_{t \to x^*} \frac{1-F(t+xf(t))}{1-F(t)} = (1+\xi x)^{-1/\xi}. \tag{4}$$

If (4) holds, then one possible f is

$$f(t) = \begin{cases} \xi t, & \xi > 0, \\ -\xi(x^* - t), & \xi < 0, \\ \frac{\int_t^{x^*} 1-F(x)\mathrm{d}x}{1-F(t)}, & \xi = 0. \end{cases} \tag{5}$$

Domain of attraction

Definition

The inverse of a nondecreasing function h is given by $h^{-1}(y) = \inf\{x : h(x) \geq y\}$.

The **tail quantile function U is the inverse of** $1/(1-F)$, meaning that $F(x) = 1 - 1/y$ and $U(y) = x$ are equivalent.

- It is non-decreasing on $[1, \infty)$ with $U(1) = x_*$ and $U(\infty) = x^*$.
- $U(n)$ is known as the **return level for the return period** n, the level that is exceeded once every n periods on average, which is often a target of inference in hydrology or climatology (more later).
- In financial risk management, $U(1/\alpha)$ of the loss distribution is a risk measure known as the α **value at risk**, which is the threshold loss value that the probability that the loss exceeds this vaue in a given time horizon is α.

Domain of attraction

Corollary[dHF06, Corollary 1.2.10]

The domain of attraction conditions can be expressed in terms of U:

$$\begin{aligned} &\lim_{t\to\infty} \frac{U(tx)}{U(x)} = x^{\xi} && \text{for Fréchet}(\xi > 0), \\ &\lim_{t\to\infty} \frac{x^* - U(tx)}{x^* - U(x)} = x^{1/\xi} && \text{for Weibull}(\xi > 0), x^* < \infty, \end{aligned}$$

Corollary[dHF06, Corollary 1.2.4]

The norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ can be found for the 3 subfamilies in terms of U and/or f:

$$\begin{cases} \xi > 0: & a_n = U(n), & b_n = 0, & F \in \mathcal{D}(\Phi_{1/\xi}), \\ \xi < 0: & a_n = x^* - U(n), & b_n = x^*, & F \in \mathcal{D}(\Psi_{-1/\xi}), \\ \xi = 0: & a_n = f(U(n)), & b_n = U(n), & F \in \mathcal{D}(\Lambda). \end{cases}$$

4.3. Gumbel Domain of attraction $\mathcal{D}(\Lambda)$

Gumbel Domain of attraction $\mathcal{D}(\Lambda)$

Definition

A distribution F with right end point x^* is a **Von Mises function** if there exists a $z_0 < x^*$ such that

$$1 - F(x) = c \exp\left\{ - \int_{z_0}^{x} \frac{1}{f(u)} du \right\} \qquad \text{for } x > z_0,$$

with $c > 0$. Here $f(u) > 0$ **is an auxiliary function with** $\lim_{x \to x^*} f'(x) = 0$.

Note that the above definition of f is asymptotically equivalent to the one given in (5).

If a distribution function F is a Von Mises function then $F \in \mathcal{D}(\Lambda)$.

Gumbel domain of attraction $\mathcal{D}(\Lambda)$

Helpful result [Resnick, 1987, Proposition 1.1]

Suppose F is absolutely continuous with negative second derivative F'' for all $x \in (z_0, x^*)$. Then if

$$\lim_{x \to x^*} \frac{F''(x)(1 - F(x))}{(F'(x))^2} = -1$$

then F **is a Von Mises function**, and $f(x) = (1 - F(x))/F'(x)$**.**

This means that if the above holds:

- we know the limit distribution is Gumbel
- we know how to calculate the normalising constants.

Gumbel domain of attraction $\mathcal{D}(\Lambda)$

Example - Standard Gaussian distribution

Since $F(x) = \Phi(x)$, $F'(x) = \phi(x)$ and $F''(x) = -x\phi(x)$, then

$$\lim_{x\to\infty} \frac{(1-F(x))F''(x)}{(F'(x))^2} = \lim_{x\to\infty} \frac{-x^{-1}\phi(x)x\phi(x)}{(\phi(x))^2} = -1,$$

and so F **is Von Mises and** $F \in \mathcal{D}(\Lambda)$**.**

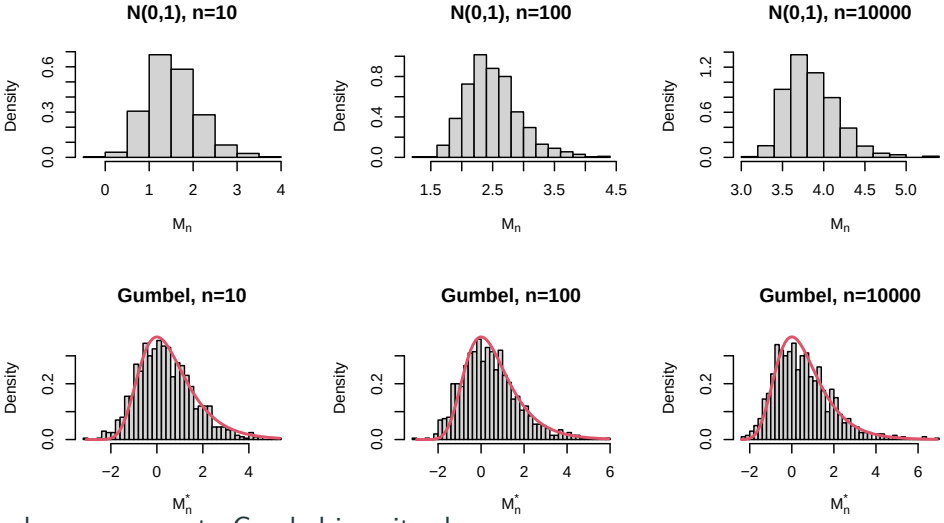
For large x, Mill's ratio (**Feller (1971)**) gives $(1-\Phi(x))/\phi(x) \sim 1/x$. Consequently,

$$f(x) = \frac{1-F(x)}{F'(x)} = \frac{1-\Phi(x)}{\phi(x)} \sim \frac{x^{-1}\phi(x)}{\phi(x)} = \frac{1}{x}.$$

Finally (and after a bit of work ...),

- $a_n = (2\log n)^{-1/2}$,
- $b_n = (2\log n)^{1/2} - 1/2(\log\log n + \log 4\pi)/(2\log n)^{1/2}$.

Gumbel domain of attraction $\mathcal{D}(\Lambda)$



Normal convergence to Gumbel is quite slow.

Gumbel domain of attraction $\mathcal{D}(\Lambda)$



Exercise

Show whether $F \in \mathcal{D}(\Lambda)$ when:

- $F = 1 - e^{-x}, x > 0$ (unit exponential distribution);
- $F = 1 - e^{-[x]}, x > 0$ (geometric distribution);
- $F = 1 - \sum_{k>x} e^{-\lambda} \frac{\lambda^k}{k!}, x > 0, \lambda > 0$ (Poisson distribution);

If $F \in \mathcal{D}(\Lambda)$, derive the norming constants $a_n > 0$ and b_n.

Hint: Check **Resnick (1987), pp.42–46** and see **pp.71–72** for details about the standard Gaussian.

4.4. Fréchet Domain of attraction $\mathcal{D}(\Lambda)$

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Fréchet domain of attraction $\mathcal{D}(\Phi)$

Definition

A measurable function $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ is **regularly varying at** ∞ with index γ ($\ell \in \mathrm{RV}_\gamma$) if

$$\lim_{x\to\infty} \frac{\ell(xu)}{\ell(x)} = u^\gamma, \quad u > 0.$$

If $\gamma = 0$ we call ℓ **slowly varying**. The above implies that if $\ell \in \mathrm{RV}_\gamma$ then $h(x) = x^{-\gamma}\ell(x) \in \mathrm{RV}_0$.

We can always represent a regularly varying function as $x^\gamma h(x)$.

Fréchet domain of attraction $\mathcal{D}(\Phi)$



Exercise

Prove the following assertions:

- $f(x) = \log(x) \in \mathrm{RV}_0$.
- If $h(x) \in \mathrm{RV}_0$ then $1/h(x) \in \mathrm{RV}_0$.
- $f(x) = 1/\log(x) \in \mathrm{RV}_0$.
- $f(x) = x^\alpha \log(1+x) \in \mathrm{RV}_\alpha$.

Fréchet domain of attraction $\mathcal{D}(\Phi)$



Solutions

- $f(x) = \log(x) \in \mathrm{RV}_0$.
  $\lim_{x\to\infty} \frac{\log(xu)}{\log(x)} = \lim_{x\to\infty} \frac{u/(xu)}{1/x} = 1$.
- If $h(x) \in \mathrm{RV}_0$ then $1/h(x) \in \mathrm{RV}_0$.
  $\lim_{x\to\infty} \frac{1/h(xu)}{1/h(x)} = \lim_{x\to\infty} \frac{h(x)}{h(xu)} = 1$ since $\lim_{x\to\infty} \frac{h(xu)}{h(x)} = 1$.
- $f(x) = 1/\log(x) \in \mathrm{RV}_0$.
  Direct by applying the first two results.
- $f(x) = x^\alpha \log(1+x) \in \mathrm{RV}_\alpha$.
  $\lim_{x\to\infty} \frac{(xu)^\alpha \log(1+xu)}{x^\alpha \log(1+x)} = u^\alpha$ by using the fact that $\lim_{x\to\infty} \frac{\log(1+xu)}{\log(1+x)} = 1$.

Fréchet domain of attraction $\mathcal{D}(\Phi)$

Helpful result [Karamata's Theorem]

Let h be a slowly varying function and locally bounded in $[x_0, \infty)$ for some $x_0 \geq 0$. Then

1. For $\alpha > -1$,

$$\int_{x_0}^{x} t^{\alpha} h(t) \mathrm{d}t \sim (\alpha + 1)^{-1} x^{\alpha+1} h(x), \quad x \to \infty,$$

2. For $\alpha < -1$,

$$\int_{x}^{\infty} t^{\alpha} h(t) \mathrm{d}t \sim -(\alpha + 1)^{-1} x^{\alpha+1} h(x), \quad x \to \infty.$$

This result becomes handy when finding the norming constants $a_n > 0$ and $b_n \in \mathbb{R}$.

Fréchet domain of attraction $\mathcal{D}(\Phi)$

Helpful result [Gnedenko, 1943]

$F \in \mathcal{D}(\Phi_\alpha)$ **if and only if** $1 - F \in \mathrm{RV}_{-\alpha}$**.** In this case $a_n = U(n)$ and $b_n = 0$

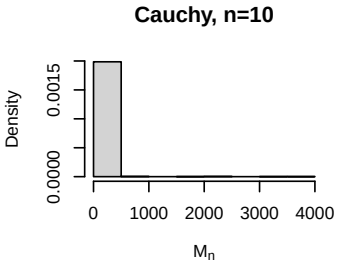
Example - Cauchy distribution

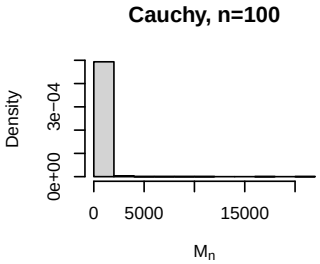
Consider the pdf $f(x) = \left(\pi(1+x^2)\right)^{-1}, x \in \mathbb{R}$. Then **as** $x \to \infty$**,** $f(x) \sim (\pi x^2)^{-1}$. By Karamata's Theorem,

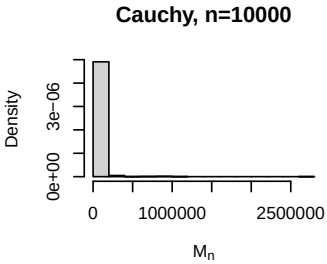
$$1 - F(x) = \int_x^\infty f(t)\mathrm{d}t \sim \int_x^\infty \pi^{-1} t^{-2} \mathrm{d}t = (\pi x)^{-1}.$$

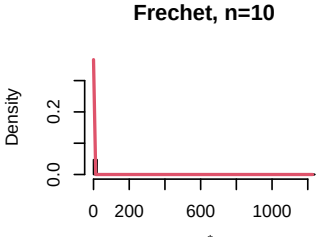
Therefore $F \in \mathcal{D}(\Phi_1)$ and a_n is such that $1 - F(a_n) = 1/n \Longrightarrow \mathbf{a_n = n/\pi}$.

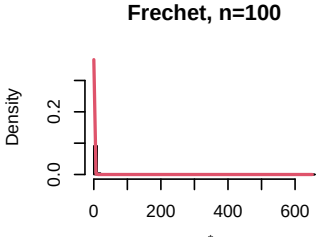
Fréchet domain of attraction $\mathcal{D}(\Phi)$

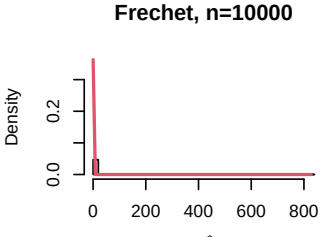












Fréchet domain of attraction $\mathcal{D}(\Phi)$



Exercise

Consider the log-Gamma distribution with pdf:

$$f(x) = \frac{\alpha^{\beta} (\log(x))^{\beta-1}}{\Gamma(\beta) x^{\alpha+1}}, \quad x > 1.$$

Show that $F \in \mathcal{D}(\Phi)$ and derive the norming constant(s).

Hint: You will need to apply Karamata's theorem on the density f.

Solution (1/4) Let's first check if $f \in \mathrm{RV}_?$.

$$\begin{aligned}\lim_{x\to\infty} \frac{f(ux)}{f(x)} &= \lim_{x\to\infty} \frac{\frac{\alpha^\beta(\log(xu))^{\beta-1}}{\Gamma(\beta)(xu)^{\alpha+1}}}{\frac{\alpha^\beta(\log(x))^{\beta-1}}{\Gamma(\beta)x^{\alpha+1}}} \\ &= \lim_{x\to\infty} \left(\frac{\log(xu)}{\log(x)}\right)^{\beta-1} \left(\frac{x}{ux}\right)^{\alpha+1} \\ &= u^{-(\alpha+1)},\end{aligned}$$

therefore $f \in \mathrm{RV}_{-(\alpha+1)}$.

Solution (2/4) We can write

$$\begin{aligned} f(x) &= x^{-(\alpha+1)} h(x) \\ &= x^{-(\alpha+1)} \frac{\alpha^{\beta} (\log(x))^{\beta-1}}{\Gamma(\beta)}. \end{aligned}$$

By Karamata's theorem,

$$\begin{aligned} 1 - F(x) &= \int_x^{\infty} t^{-(\alpha+1)} h(t) \mathrm{d}t \\ &= -\left([-(\alpha+1)]+1\right)^{-1} x^{-(\alpha+1)+1} h(x) \\ &= \frac{1}{\alpha} x^{-\alpha} \frac{\alpha^{\beta} (\log(x))^{\beta-1}}{\Gamma(\beta)} \\ &= \frac{x^{-\alpha} \alpha^{\beta-1} (\log(x))^{\beta-1}}{\Gamma(\beta)} \end{aligned}$$

Solution (3/4) Consequently

$$\lim_{x\to\infty}\frac{1-F(xu)}{1-F(x)} = \lim_{x\to\infty}\frac{(xu)^{-\alpha}\log(xu)^{\beta-1}}{x^{-\alpha}\log(x)^{\beta-1}}$$
$$= \lim_{x\to\infty} u^{-\alpha}\left(\frac{\log(xu)}{\log(x)}\right)^{\beta-1} = u^{-\alpha}.$$

This means that $1-F \in \mathrm{RV}_{-\alpha}$ which is equivalent to $F \in \mathcal{D}(\Phi_\alpha)$. The constant a_n can then be chosen as

$$\frac{1}{n} = 1-F(a_n) \sim \frac{a_n^{-\alpha}\alpha^{\beta-1}(\log(a_n))^{\beta-1}}{\Gamma(\beta)},$$

and consequently

$$a_n^{-\alpha} \sim \frac{\Gamma(\beta)}{n\alpha^{\beta-1}(\log(a_n))^{\beta-1}}.$$

Solution (4/4) Thus we have

$$-\alpha \log(a_n) \sim \log(\Gamma(\beta)) - \log(n) - (\beta - 1)\log(\alpha) - (\beta - 1)\log(\log(a_n)),$$

and $\log(a_n) \sim \frac{1}{\alpha}\log(n)$. Finally we obtain

$$\begin{aligned} a_n &\sim \left(\frac{\Gamma(\beta)}{n\alpha^{\beta-1}(\log(a_n))^{\beta-1}} \right)^{-1/\alpha} \\ &= \left(\frac{\Gamma(\beta)}{n\alpha^{\beta-1}\left(\frac{\log(n)}{\alpha}\right)^{\beta-1}} \right)^{-1/\alpha} \\ &= \left(\frac{\Gamma(\beta)}{n\left(\log(n)\right)^{\beta-1}} \right)^{-1/\alpha} \\ &= \left(\Gamma(\beta)^{-1} n \left(\log(n)\right)^{\beta-1} \right)^{1/\alpha} \end{aligned}$$

4.5. Weibull Domain of attraction $\mathcal{D}(\Psi)$

Weibull Domain of attraction $\mathcal{D}(\Psi)$

Recall: if X is such $F_X \in \mathcal{D}(\Phi)$ then $Y = -X^{-1}$ with $F_Y \in \mathcal{D}(\Psi)$ (**previous exercise**)

Therefore we may expect $\mathcal{D}(\Phi)$ and $\mathcal{D}(\Psi)$ are closely related.

Helpful result [Gnedenko, 1943]
$F \in \mathcal{D}(\Psi_\alpha), \alpha > 0$, if and only if $x^* < \infty$ and $1 - F(x^* - 1/x) \in \mathrm{RV}_{-\alpha}$. In this case $a_n = x^* - U(n)$ and $b_n = x^*$.

Note that the above implies that $1 - F(x^* - 1/x) = x^{-\alpha} h(x)$ for some slowly varying function h.

Example - Uniform distribution on $(0, 1)$
Obviously, $x^* = 1$ and $1 - F(1 - 1/x) = x^{-1}$. Therefore $F \in \mathcal{D}(\Psi_1)$. Noting that $F^{\leftarrow}(y) = y$, the norming constants are $a_n = 1 - F^{\leftarrow}(1 - 1/n) = n^{-1}$and $b_n = 1$.

Weibull Domain of attraction $\mathcal{D}(\Psi)$



Exercise

Consider the Beta distribution with pdf:

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1},\ 0 < x < 1,\ a, b > 0.$$

Show that $F \in \mathcal{D}(\Psi)$ and derive the norming constant(s).

Hint: You will need to apply Karamata's theorem.

Solution (1/3)

Obviously $x^* = 1$. Note that

$$f(1-1/x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-1/x)^{a-1}x^{1-b},$$

and thus

$$\begin{aligned}\lim_{x\to\infty}\frac{f(1-1/(ux))}{f(1-1/x)} &= \lim_{x\to\infty}\frac{(1-1/(ux))^{a-1}(ux)^{1-b}}{(1-1/x)^{a-1}x^{1-b}}\\ &= \lim_{x\to\infty}\left(\frac{1-1/(ux)}{1-1/x}\right)^{a-1}\left(\frac{ux}{x}\right)^{1-b}\\ &= u^{1-b},\end{aligned}$$

therefore $f(1-1/x) \in \mathrm{RV}_{1-b}$.

Solution (2/3) We can write

$$\begin{aligned} f(1-1/x) &= x^{b-1}h(x) \\ &= x^{b-1}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-1/x)^{a-1}. \end{aligned}$$

By Karamata's theorem,

$$\begin{aligned} 1-F(1-1/x) &= \int_{1-1/x}^{1} f(t)\mathrm{d}t \\ &= \int_{x}^{\infty} f(1-1/t)t^{-2}\mathrm{d}t \\ &= \int_{x}^{\infty} t^{-(b+1)}h(t)\mathrm{d}t \\ &\sim b^{-1}x^{-b}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-1/x)^{a-1}. \end{aligned}$$

Solution (3/3) Finally we have

$$\lim_{x\to\infty} \frac{1-F(1-1/(ux))}{1-F(1-1/x)} = \lim_{x\to\infty} \left(\frac{xu}{x}\right)^{-b} \left(\frac{1-1/(xu)}{1-1/x}\right)^{a-1}$$
$$= u^{-b}.$$

This means that $1-F(1-1/x) \in \mathrm{RV}_{-b}$ which is equivalent to $F \in \mathcal{D}(\Psi_b)$. The constant a_n can then be chosen as

$$\frac{1}{n} = 1-F(1-a_n) \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b+1)} a_n^b$$

and consequently

$$a_n \sim \left(\frac{\Gamma(a)\Gamma(b+1)}{n\Gamma(a+b)}\right)^{1/b}.$$

References



Sources and recommending reading

1. **[dHF06]** de Haan, L. & Ferreira, A. (2006) Extreme Value Theory: An Introduction. New York: Springer.
2. **[C01]** Coles, S. (2001) An Introduction to Statistical Modeling of Extreme Values. Springer. Chapter 1-3.