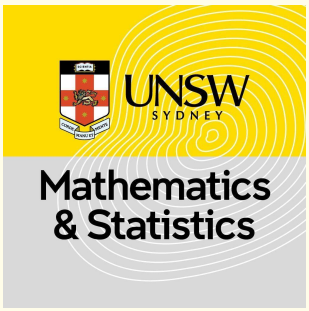


MATH5805
A Modern Introduction to Extreme Value Theory
Term 1, 2025

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Week 10 - Tutorial Solutions

Exercise 1

Show that the marginal distribution if the multivariate extreme value distribution G has Fréchet margins if the spectral measure H' satisfies

$$\int_{\mathbb{W}} \omega_j \, \mathrm{d}H'(\boldsymbol{\omega}) = 1,$$

for all $j = 1, \ldots, d$.

Solution:
The moment condition implies that $H'(\mathbb{W}) = d$. Let $H := H'/H'(\mathbb{W})$. Then

$$\begin{aligned} G_0(\boldsymbol{y}) &= \exp\left(-V(\boldsymbol{y})\right) \\ &= \exp\left(-d \int_{\mathbb{W}} \bigvee_{k=1}^{d} \frac{\omega_k}{y_k} \, H(\mathrm{d}\boldsymbol{\omega})\right) \\ &= \exp\left(-\int_{\mathbb{W}} \bigvee_{k=1}^{d} \frac{\omega_k}{y_k} \, H'(\mathrm{d}\boldsymbol{\omega})\right) \end{aligned}$$

To obtain the marginal distribution along the i-th component, let $y_i = +\infty$ for all $i \neq j$. Using the convention that $1/\infty = 0$, we have that

$$\begin{aligned} G_j(y_j) &= G_0(+\infty, \ldots, y_j, \ldots, +\infty) \\ &= \exp\left(-\int_{\mathbb{W}} \frac{\omega_j}{y_j} \vee \left(\bigvee_{i \neq j} \frac{\omega_i}{y_i}\right) \, H'(\mathrm{d}\boldsymbol{\omega})\right) \\ &= \exp\left(-\frac{1}{y_j} \int_{\mathbb{W}} \omega_j \, H'(\mathrm{d}\boldsymbol{\omega})\right) \\ &= \exp\left(-\frac{1}{y_j}\right), \end{aligned}$$

which is simply the unit Fréchet distribution along the j-th margin.

Exercise 2

Let R be the tail dependence function. Show that

$$R(\boldsymbol{z}) = d \int_{\mathbb{W}} \min_{k=1,\ldots,d} \frac{\omega_k}{z_k} \, \mathrm{d}H(\boldsymbol{\omega}).$$

Solution:
Recall that we constructed the Poisson Process

$$N_n := \left\{ \left(\frac{i}{n+1}, \frac{Z_{i1}}{n}, \ldots, \frac{Z_{id}}{n} \right), i = 1, \ldots, n \right\},$$

and that N_n converges to a non-homogenous Poisson process N with intensity measure $\Lambda = m \times \eta$, where m is the Lebesgue measure and

$$\eta\left(A_{\boldsymbol{z}}\right) = \eta\left\{ \boldsymbol{v} \in \mathbb{R}_+^d \setminus \{\boldsymbol{0}\} : v_1 > z_1, \text{ or } v_2 > z_2, \ldots, \text{ or } v_d > z_d \right\} \equiv V(\boldsymbol{y}).$$

In contrast, the set

$$\mathcal{F}_{\boldsymbol{z}} = \{ \boldsymbol{v} \in \mathbb{R}_+^d \setminus \{\boldsymbol{0}\} : v_k > z_i \text{ for all } k = 1, \ldots, d \}.$$

defines joint failure for all components. We hence have that

$$\begin{aligned}
\eta\left(\mathcal{F}_{\boldsymbol{z}}\right) &= \psi\left(\left\{ Q\left(\boldsymbol{v} \in \mathbb{R}_+^d \setminus \{\boldsymbol{0}\} : v_k > z_k, \text{ for all } k = 1, \ldots, d \right) \right\} \right) \\
&= \psi\left(\left\{ (r, \boldsymbol{\omega}) \in \mathbb{R}_+ \times \mathbb{W} : r > \bigvee_{k=1}^{d} \frac{z_k}{\omega_k} \right\} \right) \\
&= \int_{\mathbb{W}} \int_{\bigvee_{k=1}^{d} \omega_k^{-1} z_k}^{\infty} \frac{1}{r^2} \, \mathrm{d}r \, \mathrm{d}H'(\boldsymbol{\omega}) \\
&= \int_{\mathbb{W}} \bigwedge_{k=1}^{d} \frac{\omega_k}{z_k} \, \mathrm{d}H'(\boldsymbol{\omega}),
\end{aligned}$$

using similar results we obtained from Exercise 2, Week 9 Tutorial. The representation is then obtained from the definition $\eta\left(\mathcal{F}_{\boldsymbol{z}}\right) = R(\boldsymbol{z})$.