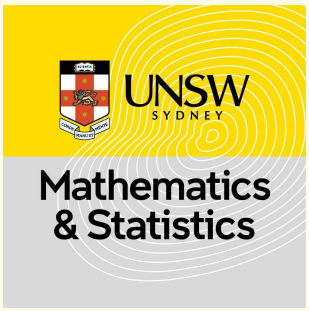


MATH5805
A Modern Introduction to Extreme Value Theory
Term 1, 2025

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Week 1 - Tutorial Solutions

Exercise 1

Consider a real-valued random variable X with distribution function F_X and probability density function f_X. Let $X_1, \dots, X_n$ denote n independent and identically distributed replicates of X.

1. Define the random variable $Y = \max(X_1, \dots, X_N)$. What are its distribution function (F_Y) and probability density function (f_Y)?

   Solution:
   The distribution function is

$$\begin{aligned} F_Y(y) &= \mathbb{P}\{Y \leq y\} \\ &= \mathbb{P}\{\max(X_1, \dots, X_N) \leq y\} \\ &= \mathbb{P}\{X_1 \leq y, \dots, X_N \leq y\} \\ &= \mathbb{P}\{X_1 \leq y\} \times \cdots \times \mathbb{P}\{X_N \leq y\} \\ &= \mathbb{P}\{X \leq y\}^n \\ &= F_X(y)^n. \end{aligned}$$

   The probability density function is given by

$$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y} F_X(y)^n \\ &= n F_X(y)^{n-1} f_X(y). \end{aligned}$$

2. Define the random variable $Z = \min(X_1, \dots, X_N)$. What are its distribution function (F_Z) and probability density function (f_Z)?

   Solution:

The distribution function is

$$\begin{aligned}
F_Z(z) &= \mathbb{P}\{Z \le z\} \\
&= \mathbb{P}\{\min(X_1, \ldots, X_N) \le z\} \\
&= \mathbb{P}\{-\max(-X_1, \ldots, -X_N) \le z\} \\
&= \mathbb{P}\{\max(-X_1, \ldots, -X_N) \ge -z\} \\
&= 1 - \mathbb{P}\{\max(-X_1, \ldots, -X_N) \le -z\} \\
&= 1 - \mathbb{P}\{-X_1 \le -z, \ldots, -X_N \le -z\} \\
&= 1 - \mathbb{P}\{X_1 \ge z, \ldots, X_N \ge z\} \\
&= 1 - \mathbb{P}\{X \ge z\}^n \\
&= 1 - (1 - F_X(z))^n
\end{aligned}$$

The probability density function is given by

$$\begin{aligned}
f_Z(z) &= \frac{\mathrm{d}}{\mathrm{d}z} F_Z(z) \\
&= \frac{\mathrm{d}}{\mathrm{d}z} (1 - (1 - F_X(z))^n) \\
&= n\,(1 - F_X(z))^{n-1}\, f_X(z).
\end{aligned}$$

3. What is the joint distribution function and probability density function of (Y, Z)?

Solution:
We are interested in

$$\begin{aligned}
F_{(Y,Z)}(y, z) &= \mathbb{P}\{Y \le y, Z \le z\} \\
&= \mathbb{P}\{\max(X_1, \ldots, X_n) \le y, \min(X_1, \ldots, X_n) \le z\}
\end{aligned}$$

Note that if $z > y$ then

$$\mathbb{P}\{\max(X_1, \ldots, X_n) \le y, \min(X_1, \ldots, X_n) \le z\} = \mathbb{P}\{\max(X_1, \ldots, X_n) \le y\},$$

thus we consider the case where $z \le y$. We have

$$\begin{aligned}
F_{(Y,Z)}(y, z) &= \mathbb{P}\{\max(X_1, \ldots, X_n) \le y, \min(X_1, \ldots, X_n) \le z\} \\
&= \mathbb{P}\{\max(X_1, \ldots, X_n) \le y\} - \mathbb{P}\{\max(X_1, \ldots, X_n) \le y, \min(X_1, \ldots, X_n) > z\} \\
&= \mathbb{P}\{X_1 \le y, \ldots, X_n \le y\} - \mathbb{P}\{z < X_1 \le y, \ldots, z < X_n \le y\} \\
&= \mathbb{P}\{X \le y\}^n - \mathbb{P}\{z < X \le y\}^n \\
&= F_X(y)^n - (F_X(y) - F_X(z))^n .
\end{aligned}$$

(Sketch a graph on (Y, Z) to see that the second line is true.) The probability density function is then

$$\begin{aligned}
f_{(Y,Z)}(y, z) &= \frac{\mathrm{d}}{\mathrm{d}z} \frac{\mathrm{d}}{\mathrm{d}y} F_{(Y,Z)}(y, z) \\
&= \frac{\mathrm{d}}{\mathrm{d}z} \left(n F_X(y)^{n-1} f_X(y) - n\,(F_X(y) - F_X(z))^{n-1}\, f_X(y) \right) \\
&= n f_X(y) \frac{\mathrm{d}}{\mathrm{d}z} \left(F_X(y)^{n-1} - (F_X(y) - F_X(z))^{n-1} \right) \\
&= n(n-1) f_X(y) f_X(z)\, (F_X(y) - F_X(z))^{n-2} .
\end{aligned}$$

4. Assume that X is in fact uniformly distributed on $[0,1]$. According to which common distributions are Y and Z distributed? Write down the joint probability density function of (Y,X).

Solution:
We have $f_X(x) = \mathbb{I}_{\{x\in[0,1]\}}$ and

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases}.$$

By construction it is known that $Y \in [0,1]$ and $Z \in [0,1]$.
Therefore, for $y \in [0,1]$ we have:

$$\begin{aligned} f_Y(y) &= nF_X(y)^{n-1} f_X(y) \\ &= ny^{n-1}, \end{aligned}$$

which is the probability density function of a Beta random variable with shape n and 1, i.e. $Y \sim \text{Beta}(n,1)$. Similarly, for $z \in [0,1]$ we have:

$$\begin{aligned} f_Z(z) &= n\left(1 - F_X(z)\right)^{n-1} f_X(z) \\ &= n\left(1-z\right)^{n-1}, \end{aligned}$$

which represents the probability density function of a Beta random variable with shape 1 and n, i.e. $Z \sim \text{Beta}(1,n)$. For $0 \le z \le y \le 1$, we have

$$\begin{aligned} f_{(Y,Z)}(y,z) &= n(n-1) f_X(y) f_X(z) \left(F_X(y) - F_X(z)\right)^{n-2} \\ &= n(n-1)\left(y-z\right)^{n-1}. \end{aligned}$$